October 18, 2006

Introduction to Green's Functions: Lecture notes[1]

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Abstract

In the present notes I try to give a better conceptual and intuitive understanding of what Green's functions are. As I hope to convey, the concept of Green's functions is very close to physical intuition, and you know already many important examples without (perhaps) being aware of it.

Aims (what I hope you will get out of these notes):
(i) know a few important examples of Green's functions,
(ii) know if a given problem can be solved by Green's functions,
(iii) write down the defining equations of a Green's functions for such problems,
(iv) know how to use Green's functions to solve certain problems.
(v) know how Green's functions are related to Fourier's method

WARNING: Beware of typos: I typed this in quickly. If you find mistakes please let me know by email.

Prologue

Green's functions provide a powerful tool to solve **linear** *problems consisting of a differential equation (partial or ordinary, with, possibly, an inhomogeneous term) and enough initial- and/or boundary conditions (also possibly inhomogeneous) so that this problem has a unique solution. The Green's function is defined by a similar problem where all initial- and/or boundary conditions are homogeneous and the inhomogeneous term in the differential equation is a delta function. If one knows the Green's function of a problem one can write down its solution in closed form as linear combinations of integrals involving the Green's function and the functions appearing in the inhomogeneities. Green's functions can often be found in an explicit way, and in these cases it is very efficient to solve the problem in this way.*

[1] I thank Andreas Minne for helpful feedback.

To give a specific example: Consider the problem to find the function $u(\mathbf{x},t)$, $\mathbf{x}\in\Omega$ *(= subset of* $\mathbb{R}^D$, $D=1,2,3$, *with boundary* $\partial\Omega$*) and* $t\geq 0$, *satisfying the PDE (= heat equation)*

$$D\Delta u - u_t = -h \quad (PDE),$$

boundary condition

$$u|_{\partial\Omega} = \alpha \quad (RV),$$

and initial condition

$$u|_{t=0} = u_0 \quad (IC)$$

for given functions $h=h(\mathbf{r},t)$, $\alpha=\alpha(\mathbf{r},t)$ *(which is defined for* $\mathbf{r}\in\partial\Omega$*), and* $u_0 = u_0(\mathbf{r})$. *Then the Green's function* G *is the solution of the similar problem*

$$D\Delta G - G_t = -\delta_{\mathbf{r}',t'} \quad (PDE'),$$

$$G|_{\partial\Omega} = 0 \quad (RV'),$$

$$G|_{t=0} = 0 \quad (IC')$$

where now all but the ODE are homogeneous, and $\delta_{\mathbf{r}',t'}$ *is the delta function localized at the spacetime point* $\mathbf{r}=\mathbf{r}'$, $t=t'$, *i.e.,* $\delta_{\mathbf{r}',t'}(\mathbf{r},t) = \delta^D(\mathbf{r}-\mathbf{r}')\delta(t-t')$. *Note that the Green's function depends on twice as many variables as u:* $G = G(\mathbf{r},t;\mathbf{r}',t')$ *(since it depends on where the delta function is localized), and thus we should write the problem determining* G *in more detail as follows,*

$$\begin{aligned} D\Delta_{\mathbf{r}} G(\mathbf{r},t;\mathbf{r}',t') - G_t(\mathbf{r},t;\mathbf{r}',t') &= -\delta^D(\mathbf{r}-\mathbf{r}')\delta(t-t') \\ G(\mathbf{r},t;\mathbf{r}',t')|_{\mathbf{r}\in\partial\Omega} &= 0 \\ G(\mathbf{r},0;\mathbf{r}',t') &= 0 \end{aligned}$$

where the $\Delta_{\mathbf{r}}$ *means that the differentiation acts on the variable* $\mathbf{r}$. *As shown in the course book, given* G *we can write the solution of our problem above as follows,*

$$u(\mathbf{r},t) = \int_0^t dt' \int_\Omega d^D r'\, G(\mathbf{r},t;\mathbf{r}',t') h(\mathbf{r}',t') +$$
$$\int_\Omega d^D r'\, G(\mathbf{r},t;\mathbf{r}',0)u_0(\mathbf{r}') - \int_{\partial\Omega} d^{n-1}S\; [\mathbf{n}\cdot\nabla_{\mathbf{r}'} G(\mathbf{r},t;\mathbf{r}',t')]\alpha(\mathbf{r}')$$

where each term on the r.h.s. accounts for one inhomogeneity in our original problem (the last integral is over the boundary of Ω, *and the normal derivative* $\mathbf{n}\cdot\nabla_{\mathbf{r}'}$ *only acts on* G*; for* $n=1$ *the boundary* $\partial\Omega$ *only consists of two points and the last integral reduced to a sum of endpoints).*

Below I give a more detailed discussion of various examples of Green's functions which you probably already know from other courses. I try to explain things in a way that the generalization of the method to other cases should be obvious. A systematic and complementary discussion can be found in our course book.

Examples you already know

I expect that most of what I discuss in the examples below is repetition for you. However, it still should be worthwhile to go through these arguments in all detail since I discuss things in a way which can be immediately adapted to other cases.

Example 1: I first recall that the **Coulomb potential** is an important example of a Green's function: as you know, the Coulomb potential $\frac{1}{4\pi|\mathbf{r}|}$ corresponds to the electric potential in three dimensional space $\mathbb{R}^3$ which is generated by a point charge sitting in the origin $\mathbf{r} = \mathbf{0}$. I now recall a mathematical characterization of the Coulomb potential: Mathematically, this point charge can be described by the charge distribution

$$\rho(\mathbf{r}) = \delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z) \tag{1}$$

where $\mathbf{r} = (x, y, z) \in \mathbb{R}^3$. Indeed, by definition of the delta function, $\delta^3(\mathbf{r}) = 0$ for $\mathbf{r} \neq \mathbf{0}$ and $\delta(\mathbf{0}) = +\infty$ such that the total charge equals 1,

$$\int_{\mathbb{R}^3} d^3\mathbf{r}\, \delta^3(\mathbf{r}) = 1.$$

We know from electrostatics that the electrical potential $V(\mathbf{r})$ generated by a charge distribution $\rho(\mathbf{r})$ obeys the **Poisson equation**,[2]

$$-\Delta V(\mathbf{r}) = \rho(\mathbf{r}) \tag{2}$$

where $\Delta V := V_{xx}+V_{yy}+V_{zz}$. We thus conclude that the Coulomb potential $V(\mathbf{r}) = \frac{1}{4\pi|\mathbf{r}|}$ is a solution of $-\Delta V(\mathbf{r}) = \delta^3(\mathbf{r})$. Obviously, if we put the point charge not in the origin but in another point $\mathbf{r}'$, then the Coulomb potential is $V(\mathbf{r}) = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$, and it obeys $-\Delta V(\mathbf{r}) = \delta^3(\mathbf{r} - \mathbf{r}')$. Now the potential depends on two arguments $\mathbf{r}$ and $\mathbf{r}'$, and to indicate this we write $V(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}')$, i.e.,

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}; \tag{3}$$

we use the symbol G since this is an example of a Green's function: *the Coulomb potential $G(\mathbf{r}, \mathbf{r}')$ above is the Green's function of the Poisson equation (2) in $\mathbb{R}^3$.* The equation determining this Green's function is obtained from the Poisson equation in (2) by choosing as inhomogeneous term a delta-function localized at an arbitrary point $\mathbf{r}'$,

$$-\Delta_{\mathbf{r}} G(\mathbf{r}, \mathbf{r}') = \delta^3(\mathbf{r} - \mathbf{r}'); \tag{4}$$

the subscript of the Laplacian is to indicate that the differentiations are to act on the $\mathbf{r}$-variable.

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[2]Strictly speaking we also should impose the boundary condition $V(\mathbf{r}) \to 0$ for $|\mathbf{r}| \to \infty$, but we will ignore this in our discussion for simplicity.

I now explain what this Green's function is good for: In general we are interested in the Poisson equation (2) for an arbitrary charge distribution $\rho(\mathbf{r})$. For point charges Q_j sitting at the points $\mathbf{r}_j$, $j = 1, 2, \ldots, N \leq \infty$, this charge distribution is

$$\rho(\mathbf{r}) = \sum_{j=1}^{N} Q_j \delta^3(\mathbf{r} - \mathbf{r}_j). \tag{5}$$

Since the potential generated by the point change $\delta^3(\mathbf{r} - \mathbf{r}_j)$ is $G(\mathbf{r}, \mathbf{r}_j)$ and the Poisson equation is linear, we can use the superposition principle to conclude that the potential generated by the charge distribution in (5) is

$$V(\mathbf{r}) = \sum_{j=1}^{N} Q_j G(\mathbf{r}, \mathbf{r}_j) = \sum_{j=1}^{N} \frac{Q_j}{4\pi |\mathbf{r} - \mathbf{r}_j|}. \tag{6}$$

Using the defining properties of the delta-function we can write this as

$$V(\mathbf{r}) = \int_{\mathbb{R}^3} d^3\mathbf{r}'\, G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}') = \int_{\mathbb{R}^3} d^3\mathbf{r}'\, \frac{\rho(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|}. \tag{7}$$

The latter equation holds true not only for charge distributions of the form as in (5) but in general: *V in (7) is the solution of (2) for (essentially) arbitrary charge distributions ρ.* To see this we write

$$\rho(\mathbf{r}) = \int_{\mathbb{R}^3} d^3\mathbf{r}'\, \delta^3(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}')$$

using a defining property of the delta function (recalling that an integral is just the limit of a sum, this can be obtained as a limit from (5)). In this latter equation we represent an arbitrary charge distribution as a linear superposition of point charges. Since $G(\mathbf{r}, \mathbf{r}')$ is the potential generated by the point change $\delta^3(\mathbf{r} - \mathbf{r}')$, we can use again the superposition principle and conclude that the potential generated by $\rho(\mathbf{r})$ is as in (7).

We can summarize the **Green' s function method** to solve the problem in (2) as follows: We first consider the simpler problem where $\rho(\mathbf{r})$ is replaced by a delta-functions $\delta_{\mathbf{r}'}(\mathbf{r}) := \delta^3(\mathbf{r} - \mathbf{r}')$ localized at the point $\mathbf{r} = \mathbf{r}'$. The solution $u_{\mathbf{r}'}$ of this latter problem is the Green's function: $G(\mathbf{r}, \mathbf{r}') = u_{\mathbf{r}'}(\mathbf{r})$. We then can write the solution (2) in closed formal as an integral as in (7).

The advantage of the method is that it is often quite easy to find the Green's function of a given problem. Moreover, there are many different problems which have the same Green's functions.

As we will see, similar statements holds true for many linear differential equation with suitable boundary and/or initial conditions.

Example 2: As a second example I consider a pendulum in the earth gravitational field driven by an external time dependent force $F(t)$. If the deviation $y(t)$ from the equilibrium position remains small we can model this system by the harmonic oscillator equation

$$\ddot{y}(t) + \omega^2 y(t) = F(t) \quad \forall t > 0; \tag{8}$$

$\dot{y}(t) = dy(t)/dt$ etc., and $\omega > 0$ For simplicity we first consider the case with homogeneous initial conditions:

$$y(0) = \dot{y}(0) = 0; \tag{9}$$

the dot means differentiation with respect to time t. I assume you know other methods to solve this problem (e.g. Laplace transformation), but I now want to illustrate how to solve it using a Green's function. Similarly as in our first example above we define the Green's function as the solution $y_{t'}(t) = G(t,t')$ of this problem where the general inhomogeneous term $F(t)$ is replaced by a delta function $\delta_{t'}(t) = \delta(t-t')$ localized at $t = t'$, i.e.,

$$G_{tt}(t,t') + \omega^2 G(t,t') = \delta(t-t') \quad \forall t,t' > 0 \tag{10}$$

together with the initial condition

$$G(0,t') = G_t(0,t') = 0. \tag{11}$$

It is then easy to see that we can write the solution of our problem in (8) as an integral as follows,

$$y(t) = \int_0^\infty dt'\, G(t,t')F(t'), \tag{12}$$

similarly as in Example 1. Indeed, it is easy to see that the initial conditions in (11) imply (9), and

$$(\partial_t^2 + \omega^2)y(t) = \int_0^\infty dt' \underbrace{(\partial_t^2 + \omega^2)G(t,t')}_{=\delta(t-t') \text{ due to (10)}} F(t') = F(t)$$

proves (8); in the first identity we interchanged integration and differentiation, and in the second we inserted (10) and used the defining property of the delta-function.

Below we first discuss the physical interpretation of the Green's function. We then present a simple method to compute it: we will find that

$$G(t,t') = \Theta(t-t')\frac{1}{\omega}\sin(\omega(t-t')) \tag{13}$$

where Θ is the Heaviside function. We finally show that the Green's function not only can be used to solve the problem above in (8) and (9) but even the more general one with inhomogeneous initial conditions.

Physical Interpretation of (13):[3] As discusses, the Green's function $G(t,t') = y_{t'}(t)$ is the solution of (8) and (9) with $F(t) = \delta(t-t')$: the pendulum is in equilibrium position and with zero velocity at time $t = 0$. In the time interval $0 < t < t'$ it is left to itself, but at time $t = t' > 0$ it receives is hit by a force pulse, after which it is left to itself again. We thus should expect that the pendulum remains in the equilibrium position until it is hit: $y_{t'}(t) = 0$ for $t < t'$. The force pulse, however, sets the pendulum in motion, and it thus should oscillate freely for $t > t'$: $y_{t'}(t) = B\sin(\omega(t-t_1))$ for some constants B, t_1. Since the force pulse sets the pendulum in motion, right after $t = t'$ is still should be in the equilibrium position: $y_{t'}(t) = 0$ for $t \to t'$. This fixes the constant t_1: $t_1 = t'$. To find the constant B one can argue that the total pulse, i.e. the integral of the external force over a tiny time interval including the pulse time, should be equal to the abrupt velocity change of the pendulum due to the pulse: $\dot{y}_{t'}(t_2) - \dot{y}_{t'}(t_1) = \int_{t_1}^{t_2} dt\, F(t)$, where $t_1 = t' - \varepsilon$ and $t_2 = t' + \varepsilon$ with $\varepsilon > 0$, $\varepsilon \to 0$. We will show below how to derive this condition mathematically, and that it gives $B = 1/\omega$. We thus have derived (13) by physical arguments: the Heaviside function accounts for the pendulum being in equilibrium before the pulse, and otherwise it is the usual motion of a free pendulum where the free constants are fixed by the pulse.

A physical interpretation of (12) is as follows: an arbitrary external force $F(t)$ acting on the pendulum can be thought of as a linear superposition of force pulses at different times $t = t'$ of strength $F(t')$:

$$F(t) = \int_0^\infty dt'\, \delta(t-t')F(t')$$

(mathematically this latter equation is just one of the defining conditions for the delta-function, of course). Since $G(t,t')$ is the response of the pendulum to the pulse $\delta(t-t')$ and the system is linear, we should expect that the total response, i.e. the actual motion of the pendulum, is just the corresponding linear superposition of the pulse responses: this is exactly what (12) says.

It is also interesting that, due to the Heaviside function in (13), $G(t,t') = 0$ for $t' > t$, and we therefore can write (12) as

$$y(t) = \int_0^t dt'\, G(t,t')F(t'), \tag{14}$$

(the upper integration limit is now t), and this has a natural physical interpretation as **causality**: the position of the pendulum at any time t can only depend on the force *before* that time.[4]

□

[3] You might want to skip this at first reading.

[4] More generally, *causality* means something like: what *is* can only depend on the past but not on the future.

We now derive (13) mathematically.

A method to solve (10) and (11):[5]

To simplify notation I write $G(t,t') = y(t)$ and suppress the t'-dependence for now. Thus we need to solve

$$\ddot{y}(t) + \omega^2 y(t) = \delta(t-t') \forall t > 0, \quad y(0) = \dot{y}(0) = 0.$$

Since $\delta(t-t') = 0$ for $t \neq t'$, $y(t)$ solves the homogeneous oscillator equation for $t \neq t'$, and we thus conclude

$$y(t) = \begin{cases} A\sin(\omega(t-t_0)) & \text{for } 0 \leq t < t' \\ B\sin(\omega(t-t_1)) & \text{for } t' > t \end{cases} \tag{15}$$

for some constants A, B, t_0, t_1 to be determined. As discussed, the physical interpretation of this is as follows: the pendulum oscillates freely, but the delta-force changes this oscillation abruptly at time $t = t'$ (it gives a "kick"). Since $y(0) = \dot{y}(0)$ we get $A = 0$ and thus $y(t) = 0$ for $t < t'$ (the value of t_1 is irrelevant now, of course); this just corresponds to causality, but now we also have a mathematical proof of this important property. To find the constants B and t_1 we now show that the effect of the delta-force the following conditions,

$$y(t'+0) = 0, \quad \dot{y}(t'+0) = 1 \tag{16}$$

where $y(t'+0) = \lim_{\varepsilon\downarrow 0} y(t'+\varepsilon)$ etc.; by $\varepsilon \downarrow 0$ we mean $\varepsilon > 0$ and $\varepsilon \to 0$. To see this we integrate our differential equation from $t = t' - \varepsilon$ to $t > t'$ in the limit $\varepsilon \downarrow 0$:

$$\int_{t'-0}^{t} ds\, (y_{ss}(s) + \omega^2 y(s)) = \int_{t'-0}^{t} ds\, \delta(s-t') = 1,$$

and since $y(t) = \dot{y}(t) = 0$ for $t < t'$ we get from this,

$$\dot{y}(t) = 1 - \omega^2 \int_{t'}^{t} ds\, y(s) \quad \forall t > 0.$$

This proves the second condition in (16). To get the first one we integrate once more and obtain

$$y(t) = (t-t') + \int_{t'}^{t} ds \int_{t'}^{s} dr\, y(r)$$

implying the first condition in (16). Thus our solution for $t > t'$ can be found by solving

$$\ddot{y}(t) + \omega y(t) = 0 \quad \forall t > t', \quad y(t') = 0, \quad \dot{y}(t') = 1.$$

[5] The method explained here is useful to remember for the quantum mechanics course since it allows to solve the Schrödinger equation $-\psi''(x) + V(x)\psi(x) = E\psi(x)$ for singular potentials $V(x) = g\delta(x-x_0)$.

This is not difficult, and we obtain:

$$y(t) = \frac{1}{\omega}\sin(\omega(t-t')).$$

We thus get $y(t) = G(t,t')$ as in (13). This concludes our computation. □

We now consider the more general problem in (8) with inhomogeneous initial conditions,

$$y(0) = y_0, \quad \dot{y}(0) = v_0 \quad y_0, v_0 \text{ real}, \tag{17}$$

where y_0 and v_0 real parameters. From previous courses you might know some method to derive the following solution of this problem:

$$y(t) = y_0\cos(\omega t) + v_0\frac{1}{\omega}\sin(\omega t) + \int_0^t ds\,\frac{1}{\omega}\sin(\omega(t-s))F(s). \tag{18}$$

It is interesting to note that

$$y(t) = y_0 G_t(t,0) + v_0 G(t,0) + \int_0^t ds\, G(t,s)F(s), \tag{19}$$

i.e., we can write the solution solely in terms of the Green's function of the problem.

This has an important interpretation: the solution of our problem

$$\ddot{y}(t) + \omega^2 y(t) = F(t), \quad y(0) = y_0, \quad \dot{y}_0(0) = v_0$$

for $t > 0$ and inhomogeneous initial conditions is identical with the solution of the problem

$$\ddot{y}(t) + \omega^2 y(t) = F(t) + v_0\delta(t) - y_0\delta'(t), \quad y(0) = 0, \quad \dot{y}_0(0) = 0$$

for all t and $y(t) = 0$ for $t < 0$: it is possible to trade the inhomogeneous initial conditions for inhomogeneous terms in the ODE.

This is no coincidence: non-trivial initial and/or inhomogeneous boundary conditions can always be accounted for by terms involving the Green's function of the system. The physical interpretation of this is that, to account for non-trivial initial conditions, we can start with the pendulum in equilibrium position, and give it an appropriate pulse at $t = 0$.

Computation of Green's functions using Fourier's method I try explain this using a simple representative example which is our well-studied model of a string: Compute the function $u = u(x,t)$, $0 < x < L$, $t > 0$ such that

$$\begin{aligned} u_{tt}(x,t) - c^2 u_{xx}(x,t) &= F(x,t) \\ u(0,t) = u(L,t) &= 0 \\ u(x,0) = \alpha(x), \quad u_t(x,0) &= \beta(x). \end{aligned} \tag{20}$$

We discussed how to solve this problem by expanding the solution in the eigenfunctions defined by the corresponding eigenvalue problem $f''(x) + \lambda f(x) = 0$, $f(0) = f(L) = 0$:

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin(k_n x), \quad k_n = n\frac{\pi}{L}, \tag{21}$$

and deriving and solving an ODE problem for the coefficients $a_n(t)$. I now show how to solve this model using the philosophy of Green's functions.

The Green's function for this model is defined as the solution $u(x,t) = G(x,x',t,t')$ of the equations above for the special case $F(x,t) = \delta(x-x')\delta(t-t')$, where $t' > 0$ and $0 < x' < L$, and $\alpha(x) = \beta(x) = 0$. We note that

$$\delta(x-x') = \sum_{n=1}^{\infty} \frac{2}{L} \sin(k_n x) \sin(k_n x')$$

(check that!), and thus by the ansatz in (21) above we get

$$\sum_n \sin(k_n x)[a_n''(t) + (ck_n)^2 a_n(t)] = \sum_n \frac{2}{L} \sin(k_n x) \sin(k_n x') \delta(t-t')$$

etc., implying

$$a_n''(t) + (ck_n)^2 a_n(t) = \frac{2}{L} \sin(k_n x') \delta(t-t'), \quad a_n(0) = a_n'(0) = 0$$

which has the solution (of you do not remember this you can derive it using the Green's function method)

$$a_n(t) = \Theta(t-t') \frac{2}{L} \sin(k_n x') \frac{1}{ck_n} \sin(ck_n(t-t')).$$

We thus get the following formula for $G(x,t,x',t') = u(x,t)$:

$$G(x,t,x',t') = \Theta(t-t') \sum_{n=1}^{\infty} \frac{2}{L} \sin(k_n x) \sin(k_n x') \frac{1}{ck_n} \sin(ck_n(t-t')). \tag{22}$$

Using that Green's function we can write the solution of our problem in (20) as follows,

$$\begin{aligned} u(x,t) = &\int_0^L G(x,t,x',0)\beta(x')dx + G_t(x,t,x',0)\alpha(x')dx \\ &+ \int_0^{\infty} dt' \int_0^L dx\, G(x,t,x',t')F(x',t'), \end{aligned} \tag{23}$$

where the t'-integral can be restricted to $0 < t' < t$ due to causality. It is instructive to convince oneselves that this answer is the same as the one one gets using Fourier's method: I recommend you do this!

The computation above can be immediately generalized to problems where the interval $\Omega = [0, L]$ is replaced by some other bounded region Ω in $D = 1, 2, 3 \ldots$ dimensions, e.g. a disc ($D = 2$) or a sphere of a cylinder ($D = 3$) or $\ldots$. In this case the functions $\sin(k_n x)$ and k_n above are to be replaced by the eigenfunctions defined by the corresponding Helmholtz equation:

$$-\Delta u_n(\mathbf{x}) = k_n^2 u_n(\mathbf{x}) \text{ in } \Omega, \text{ plus boundary conditions,}$$

where now D can stand for several integers, and the integrals $\int_0^L dx'$ are replaced by $\int_\Omega d^D x'$. It is instructive to write everything out in one other example for Ω. The key result needed is that

$$\delta^D(\mathbf{x} - \mathbf{x}') = \sum_n \frac{1}{||u_n||^2} u_n(\mathbf{x}) \overline{u_n(\mathbf{x}')}$$

where $||u_n||^2 = \int_\Omega d^D x \, |u_n(\mathbf{x})|^2$ — note that this is just a nice way of writing that it is possible to expand any "nice" function as generalized Fourier series using these functions u_n.

Green's functions: generalities

Consider a problem

$$\mathcal{L}u = h, \quad B_1(u) = \alpha_1, \quad B_2(u) = \alpha_2, \ldots B_N(u) = \alpha_N \tag{24}$$

for some function $u = u(\mathbf{x})$, $\mathbf{x}$ in some subset Ω of $\mathbb{R}^D$ for some D (where one of the variables can be time), $\mathcal{L} = \mathcal{L}_x$ some linear differential operator acting on the variables x, and the B_j defining linear initial and/or boundary conditions.

Then the corresponding Green's function $G = G(x, x')$ is defined as solution of the following problem,

$$\mathcal{L}_x G(x, x') = \delta^D(x - x'), \quad B_1(G) = B_2(G) = \ldots = B_N(G) = 0$$

(if the the B_j involve differentiations they are to act on the variables x!) where $\delta^D(x - x')$ is the delta function localized at $x' \in \Omega$. If all initial and/or boundary conditions are homogeneous: $\alpha_j = 0 \; \forall j$, the the solution of the problem is

$$u(x) = \int_\Omega d^D x \, G(x, x') h(x').$$

Otherwise one also has to add integrals involving G and α_j, over the 'boundary regions' where α_j is defined. To find the precise form of these boundary contributions in general is somewhat tricky, but several important examples are derived in the course book (one way for accounting for an inhomogeneity α_j is to move it from the boundary/initial condition to the differential equation: this is always possible).