Exempel variationsräkning 1, 5A1305 Fysikens matematiska metoder, ht06.

Sorry for the mixture of Swedish and English — I hope I have used the same language at least within the same problem:-)
Problems with a star have an additional hint ("ledning").

1. (Brachistochron problem) En partikel glider friktionsfritt under tyngdkraftens inverkan längs kurvan $y = y(x)$ i det vertikala xy-planet, mellan punkterna A och B.

   Sök den kontinuerligt deriverbar kurva längs vilken partikeln rör sig från A till B på kortaste tid.

2. (Såphinnan) Låt $y = y(x)$ vara en kurva som går mellan punkterna (x_1, y_1) och (x_2, y_2) i xy-planet. Genom att rotera kurvan kring x-axel genereras en rotationsyta S. Bestäm $y(x)$ så att arean av S blir extremal. När är arean minimal?

3. Energitätheten i ett elektrostatiskt fält $\mathbf{E}$ ges av $\frac{1}{2}\varepsilon E^2$ där $\mathbf{E} = -\nabla\Phi$ och $\varepsilon > 0$ är dielektricitetskonstanten. Bestäm elektriska potentialen $\Phi(\mathbf{r})$ så att energin från fältet inom sfären $V : |\mathbf{r}| \leq R$ blir så liten som möjlig.

4. * En homogen tunn böjlig kedja med massan m och längden ℓ är uppgängd mellan punkterna A: $(x, y) = (0, 0)$ och $B : (x, y) = (a, 0)$ $(a < 0)$. Kedjan påverkas av ett homogent gravitationsfält med tyngdaccelerationen g. Sök kedjans jämviktskurva!

5. (Dido's problem) Bestäm den slutna kurva med given längd ℓ som omsluter den största arean.

6. En "Fata morgana" är ett optiskt fenomen som upptrder då atmosfårens brytningsindex varierar med höjden och där objekt och områden bortom horisonten blir synliga. Använd Fermats princip för att ge an förklaring.

   Ledning: Fermats princip säger att ljuset följer den väg för vilken $\int n\,\mathrm{d}s$ antar ett extremvärde, där n är bryttningsindex och $s = \int \mathrm{d}s$ bågländen. Anta att

   $$n(x, y) = n_0(1 - ay)$$

   där y-axeln är riktat uppåt, x-axeln är parallel med jordens yta, och $n_0 > 0$, $a > 0$ är konstanter. Bestäm ljusets bankurvan $y(x)$ om $y(0) = 0$ och $y'(0) = b > 0$.

7. Bestäm kortaste vägen på koniska ytan $r = -az$, $z \leq 0$, mellan punkterna $z = -h$, $\phi = 0$ och $z = -h$, $\phi = \pi/2$ där (r, ϕ, z) är cylinderkoordinater och $h > 0$.

8. En partikels bana beskrivs av polära koordinater r och ϕ. Beloppet av partikelns hastighet beror av läget enligt

   $$v = v_0(r/R)^2$$

med konstanter $v_0 > 0$ och $R > 0$. Sök kurvan längs vilken partikels förflyttar sig från punkten $r = R$, $\phi = \pi/2$ till linjen $\phi = 0$ på kortast möjlig tid. Beräkna även den kortaste tiden samt veriferiera att den är mindre än tiden längs kurvan $r = R$, $0 \le \phi \le \pi/4$.

9. Då vattnet i en rak cirkulär cylinder med radie a roterar med winkelhastigheten ω runt cylinderaxeln ormer sig ytan så at den potentiella energien U i det roterande systemet blir minimal. Bitraget $\mathrm{d}U$ från volymelementet $\mathrm{d}V$ är

$$U = \rho(gz - \frac{\omega^2 r^2}{2})\mathrm{d}V.$$

Bestäm vattenytans höjd över botten $h(r)$, om den totala vattenvolymen antas vara V_0.

10. * A homogeneous beam of length ℓ is fixed in one end and unconstrained in the other, and it is bend down under the influence of gravitation. We can assume that the one dimensional function $u(x)$ describing the beam's deviation from its tension-less, straight state minimizes the total potential energy

$$U[u] = \int_0^\ell \left(\frac{EJ}{2}(u''(x))^2 - \rho g A u(x) \right) \mathrm{d}x$$

with $g = 9.81\ldots$ m/s^2 the gravitation constant and E, J, A and ρ material constants (Young's modulus of elasticity, moment of inertia, cross section area, and mass density).

Determine the shape $u(x)$ of the beam.

11. Bestäm det minsta möjliga värdet som integralen

$$\int_{x^2+y^2\le 1} \left(u_x(x,y)^2 + u_y(x,y)^2\right) \mathrm{d}x\mathrm{d}y$$

kan antar. Funktionen $u(x,y)$ har kontinuerliga andraderivator samt uppfyller randvillkoren $u = 0$ på circeln $x^2 + y^2 = 1$. Vidare gäller att

$$\int_{x^2+y^2\le 1} u(x,y)^2 \,\mathrm{d}x\mathrm{d}y = 1.$$

Hints *WARNING: I typed this in quickly: Please let me know if you find typos!*

1. The velocity of the particle is $v = \frac{ds}{dt}$ where $ds = \sqrt{1 + y'(x)^2}$, and v can be computed from energy conservation: $mv^2/2 - mgy = const. = 0$, where $(0, 0)$ corresponds the initial point A where $v = 0$, and the y-axis points downwards.

   One thus needs to minimize the time $T = \int_{t_A}^{t_B} dt$ which, after some computations (using $dt = ds/v$), becomes

   $$T[y] = \frac{1}{2g} \int_0^b \underbrace{\frac{\sqrt{1 + y'(x)^2}}{\sqrt{y(x)}}}_{F(y(x),y'(x))} dx$$

   where $b, y(b)$ corresponds to the end point B.

   From $F - y' \frac{\partial F}{y'} = const.$ one finds after some computations

   $$y(x) = \frac{C}{1 + y'(x)^2}$$

   for some constant C. Introducing a paramtrizations $y(\theta)$ and $x(\theta)$ from $y'(x) = \cot(\theta)$ one gets $y = C \sin^(\theta)$ and

   $$x = C(\theta - \frac{1}{2} \sin(2\theta)).$$

   This solution can be written in a nicer way using $\phi = 2\theta$ and $A = C/2$.

2. Extremize

   $$A = \int_{x_1}^{x_2} 2\pi y(x) \sqrt{1 + y'(x)^2} dx$$

   where the initial and end points $(x_1, y_1 = y(x_1))$ and $x_2, y_2 = y(x_2)$ are fixed.

3. Minimize

   $$\int_{r \le R} d^3x \, \nabla\Phi(\mathbf{x})^2$$

   so that Φ is arbitrary at the boundary of the sphere. This gives

   $$\Delta\Phi = 0, \quad \hat{\mathbf{n}} \cdot \nabla\Phi|_{|\mathbf{r}|=R} = 0.$$

4. The potential energy of the chain

   $$I[y] = \rho g \int_a^b y(x) \sqrt{1 + y'(x)^2} dx$$

   should be minimized, and its length

   $$L[y] = \int_a^b \sqrt{1 + y'(x)^2} dx$$

   is fixed; $x = a$ and b correspond to the end points of the chain and $y(x)$ is its form.

   The problem can be solved my minimizing $I[y] - \lambda(L[y] - L_0)$ where L_0 is the length of the chain and λ a Lagrange multiplier.

5. It is wise to search for the curve Cin paramter representation $x(t), y(t)$ with $t_0 \leq t \leq t_1$.

   The area enclosed by the curve C is

$$I = \oint_C (xdy - ydx)$$

   (Green's theorem) and should be maximized, and the length of C

$$L = \oint_C \sqrt{dx^2 + dy^2}$$

   is fixed. It is wise to assume a paramter representation $x(t), y(t)$ for the curve where $t_0 \leq t \leq t_1$. Then

$$I[x, y] = \int_{t_0}^{t_1} (x(t)y'(t) - y(t)x'(t))dt, \quad L = \int_{t_0}^{t_1} \sqrt{x'(x)^2 + y'(x)^2}dt,$$

   and we should minimze $I - \lambda(L - L_0)$ where λ is a Lagrange multiplier and L_0 the length of the curve; we are looking for a closed curve, i.e., $(x(t_0), y(t_0)) = (x(t_1), y(t_1))$.

   The resulting Euler-Lagrange differential equations are

$$y'(t) = -\lambda \frac{d}{dt} \frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}$$

$$x'(t) = \lambda \frac{d}{dt} \frac{y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}.$$

   This gives

$$y(t) - y_0 = -\lambda \frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}, \quad x(t) - x_0 = \lambda \frac{y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}},$$

   which implies

$$(x - x_0)^2 + (y - y_0)^2 = \lambda^2.$$

6. Assume that the light ray is described by a curve $y(x)$ with the beginning- and end points fixed in $x = 0, y = 0$ and $x = \ell, y = y_1$, and show that there is such a function minimizing

$$I[y] = \int_0^{\ell} (1 - ay(x))\sqrt{1 + y'(x)^2}dx$$

   and such that $y_1 = 0$: a light ray connects then two points on the ground which are a distance ℓ apart.

   The first integral of the Euler-Lagrange equation of the problem is is

$$\frac{1 - ay(x)}{\sqrt{1 + y'(x)^2}} = C.$$

7. Use cylinder coordinates and determine the path $z(\phi)$ which minimizes

$$\int_0^{\pi/2} \sqrt{a^2 z(\phi)^2 + (1+a^2) z'(\phi)^2} d\phi$$

and which goes through the beginning- and end points.

8. Minimize

$$T[r] = \frac{R^2}{v_0} \int_0^{\pi/4} \underbrace{\frac{\sqrt{r(\phi)^2 + r'(\phi)^2}}{r(\phi)^2}}_{=F(r(\phi), r'(\phi))} d\phi$$

such that $r(\pi/4) = R$, $\partial F/\partial r'|_{\phi=0} = 0$.

9. Minimize the potential energy of the water

$$U[h] = \int_0^a drr \int_0^{2\pi} d\phi \int_0^{h(r)} dz\, \rho(gz - \frac{\omega^2 r^2}{2})$$

such that the volume

$$V[h] = \int_0^a drr \int_0^{2\pi} d\phi \int_0^{h(r)} dz$$

is constant; $h(r)$, $0 \leq r \leq a$, describes the shape of the water, and r, ϕ, z are cylinder coordinates.

Thus find the function h minimizing $U - \lambda(V - V_0)$ where $h(0)$ and $h(a)$ are free; λ is a Lagrange multiplier, and V_0 the fixed volume.

10. Use variational calculus to derive the ODE and the correct boundary conditions. Since the beam is fixed at the end $x = 0$ we only allow variational functions $u(x)$ such that $u(0) = u'(0) = 0$, and since the beam is not constrained at the other end $x = \ell$, $u(\ell)$ and $u'(\ell)$ are allowed to be arbitrary.

The Euler-Lagrange eqs. and boundary conditions are

$$u'''' - \frac{\rho g A}{EJ} = 0,$$

and $u(0) = u'(0) = u''(\ell) = u'''(\ell) = 0$.

11. The Euler-Lagrange equations are $-\Delta u = \lambda u$ where λ is a Lagrange multiplier and $u|_{|\mathbf{r}|=1} = 0$. All solutions of this problem are

$$C J_n(j_{n,s} r) \sin(n\phi - \phi_0)$$

with a normalization constant C fixed by the constraint, and $\lambda = j_{n,s}^2$; $j_{n,s}$ are the zeros of the Bessel function J_n.

By partial integration and using thee Euler-Lagrange equations and the boundary conditions one finds that

$$\int_{x^2+y^2} (u_x^2 + u_y^2) dxdy = \lambda \int_{x^2+y^2} u^2 dxdy = \lambda.$$

Solutions

1. One paramter representation of the solution is
$$y = A(1 - \cos(\phi)), \quad x = A(\phi - \sin(\phi)), \quad A > 0$$
with the real paramter ϕ.

2. $$y(x) = \frac{1}{p} \cosh(p(x - x_0))$$
where the paramters p and x_0 are determined by the conditions that $y(x_i) = y_i$ for $i = 1, 2$.

3. $\Phi(\mathbf{x}) = C$ (constant), as could have been guessed without computation.

4. $$y(x) = C_1 \cosh(x/C_1 + C_2) - \lambda$$
where the constants $C_{1,2}$ and λ are determined by the location of the endpoints and L_0.

5. Circles.

6. The extremizing functions are
$$y(x) = \frac{1}{a} - C_1 \cosh(x/C_1 + C_2)$$
with constants $C_{1,2}$ determined by the beginning- and endpoints of the light ray: $C_2 = -\ell/2$, $C_1 = 1/a$.

7. $$z(\phi) = -\frac{h \cos(\frac{a\pi}{4\sqrt{1+a^2}})}{\cos(\frac{a}{\sqrt{1+a^2}}(\phi - \pi/4))}.$$

8. $r(\phi) = R\sqrt{2}\cos(\phi)$.

9. $$h(r) = \frac{V_0}{\pi a^2} + \frac{\omega^2}{2g}(r^2 - a^2/2).$$

10. $$u(x) = \frac{\rho g A}{24EJ} x^2(x^2 - 4\ell x + 6\ell^2).$$

11. $u = j_{0,1}^2 = 5.783\ldots$, where $j_{0,1} = 2.40482\ldots$ is the smalles non-zero zero of the Bessel function J_0.