1. Let $M$ be the 2-dimensional torus $S^1 \times S^1$. Construct a differentiable structure on $M$ using an atlas consisting of two open sets.

2. The standard spherical coordinates $(\theta, \phi)$, with $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, on the unit sphere $S^2$ do not suffice to define a differentiable structure. (Why?) Find a 'minimal modification', in terms of two coordinates charts, to make $S^2$ to a manifold.

3. The group $SL(2, \mathbb{R})$ of real $2 \times 2$ matrices with determinant equal to 1 is a manifold. How?

4. The unit sphere $S^3$ can be thought of as the group $SU(2)$ of unitary complex $2 \times 2$ matrices with determinant $= 1$. Using this fact show that the tangent bundle $TS^3$ can be identified as the Cartesian product $\mathbb{R}^3 \times S^3$.

5. Check the relations

$$[X, fY] = f[X, Y] + (X \cdot f) Y \quad \text{and} \quad [fX, Y] = f[X, Y] - (Y \cdot f) X$$

for a smooth function $f$ and a pair of vector fields $X, Y$ on a manifold.

6. Let $M$ be the manifold of real nonsingular $n \times n$ matrices. For each real $n \times n$ matrix $X$ we define a flow $h^X_t$ on this manifold by $h^X_t(g) = e^{-tX}g$, with ordinary matrix multiplication. This flow defines a vector field $\hat{X}$ on $M$ as usual and for a smooth function $f$ on $M$

$$(\hat{X}.f)(g) = \frac{d}{dt} f(h^X_t(g)) |_{t=0}.$$ 

Show that the commutator $[\hat{X}, \hat{Y}]$ of vector fields corresponds to the commutator of matrices $[X, Y]$, i.e. $[\hat{X}, \hat{Y}] = [X, Y]$. 