1. Prove the commutation relations

\[ [M_{ij}, M_{kl}] = \delta_{jk} M_{il} + \delta_{il} M_{jk} - \delta_{ik} M_{jl} - \delta_{jl} M_{ik}. \]

for the matrices \( M_{ij} = \frac{1}{4} [\gamma_i, \gamma_j] \). (Euclidean metric)

2. Show that the unit sphere in any dimension is a spin manifold. Hint: Think of \( \text{Spin}(n) \) as a \( \mathbb{Z}_2 \) principal bundle over \( \text{SO}(n) \). The transition function of the frame bundle of \( S^n \) defines by pull-back a \( \mathbb{Z}_2 \) bundle over the equator.

3. Compute the 8-form part of the Chern character \( \exp \left( F / 2\pi \right) \) of a real vector bundle in terms of the Pontrjagin classes.

4. Write down explicitly the Dirac operator on the unit sphere \( S^2 \), coupled to the vector potential of the monopole bundle.

5. Let \( D \subset \mathbb{R}^n \) be the unit disk and denote by \( \mathcal{A} \) the set of smooth 1-forms on \( D \) with values in the Lie algebra \( g \) of a compact Lie group \( G \). Let \( p \) be a fixed point on the boundary of \( D \). a) Show that for each \( A \in \mathcal{A} \) there is a unique smooth \( g = g_A : D \to G \) such that \( g_A(p) = 1 \) and \( A' = \text{ad}_{g^{-1}}(A) + g^{-1} dg \) is in the radial gauge, i.e., the radial component \( x_k A_k = 0 \). b) In finite dimensions, if a Lie group \( G \) acts smoothly and freely on a manifold \( M \) then the set of orbits \( M/G \) has a natural differentiable structure making it into a smooth manifold. The same can be shown to be true in the case of \( \mathcal{A}/\mathcal{G} \), where \( \mathcal{G} \) is the group of smooth maps \( f : D \to G \) such that \( f = 1 \) on the boundary of \( D \). Let now \( \Omega G \) be the group of smooth contractible maps \( f \) from \( S^{n-1} = \partial D \) to \( G \) such that \( f(p) = 1 \). Define a map \( \mathcal{A} \to \Omega G \) by \( A \mapsto g_A|_{S^{n-1}} \). Show that this is a homotopy equivalence. Hint: Parametrize the potentials \( A \) as pairs \( (A', g_A) \) where \( A' \) is in the radial gauge. The set of potentials in radial gauge is contractible.