The MIT bag model and the Skyrme model

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The MIT bag model and the Skyrme model are presented and their capability as nucleon models is discussed. The quark model is resumed also, to motivate the nucleon models and their basic assumptions.

INTRODUCTION

In the 1950’s and 60’s when particle accelerators made it possible to access higher and higher energy regions the number of known elementary particles increased dramatically. They were actually so many that one could not really believe anymore that there are so many elementary particles. Still it was a great surprise when around 1968 electron scattering experiments at the Stanford Linear Accelerator Center (SLAC) gave the first hints that particles like the proton have an inner structure, cf. fig. 1. The quark model was suggested earlier, in 1964, but it took some time before the model was widely accepted.

THE QUARK MODEL

The quark model dates back to 1964 when Gell-Mann and Zweig independently suggested that all known hadrons could be described as bound states of only three fundamental spin-1/2 particles and their corresponding antiparticles. Gell-Mann named these particles quarks. According to the model, quarks are fermions with an electric charge of either $\frac{2}{3}$ or $\frac{1}{3}$ of the electron charge. The antiquarks have opposite charge ($\frac{-2}{3}$ and $\frac{-1}{3}$). As we know now there are not only three but six different types, so called flavors, and six corresponding antiquarks.

![Figure 1: Experimental results from (a) Rutherford scattering and (b) deep inelastic electron-proton scattering.](image)

Similar to the leptons they occur in generations, denoted by

$$\left( \begin{array}{c} u \\ d \\ c \\ s \\ t \\ b \end{array} \right) \left( \begin{array}{c} \pi \\ \eta \\ \eta' \end{array} \right) \left( \begin{array}{c} e \\ \mu \\ \tau \end{array} \right) \left( \begin{array}{c} \bar{u} \\ \bar{d} \\ \bar{c} \\ \bar{s} \\ \bar{t} \\ \bar{b} \end{array} \right).$$

For mass and charge of each quark see table I.

Contrary to other fermions, quarks have an additional quantum number called color. It can be either red ($r$), green ($g$) or blue ($b$) for quarks and antired ($\bar{r}$), antigreen ($\bar{g}$) or antiblue ($\bar{b}$) for antiquarks, respectively. Quarks interact with each other via gluons, the gauge particles of the strong interaction. Gluons are massless particles with spin 1 which carry color themselves. Therefore they interact not only with quarks but with other gluons as well. The color interaction is described by the theory of Quantum Chromodynamics (QCD).

As mentioned before all hadrons can be seen as bound states of quarks and antiquarks. To be more precise, quarks are confined to these states. A free single quark does not exist according to the model. The hadrons have to be colorless, which means that they have to contain either each one of the colors red, green and blue once or a color and the corresponding anticolor. Therefore hadrons consist of either three quarks (baryons, $qqq$), three antiquarks (antibaryons, $\bar{q}\bar{q}\bar{q}$) or a quark and an antiquark (mesons, $q\bar{q}$).

A meson is the combination of a quark and an antiquark and is therefore a boson with charge $\pm 1$ or 0. The pion ($\pi^\pm$) for instance consists of $u$ and $\bar{d}$. Baryons contain three quarks and are therefore fermions with charge $+2$, $\pm 1$ or 0. Two well known types are the proton that contains two up and one down quarks ($uud$) and the neutron that contains two down and one up quarks ($udd$).
NUCLEON MODELS

The deeper mechanism of bound quark states and quark confinement is still not completely understood. The quark model restricts valid quark states to so called colorless or color singlet states and does not allow free quarks. The MIT bag model [2, 3] is a rather straightforward approach to describe these properties by choosing the right boundary conditions for the quark wave functions, whereas the Skyrme model [4], that was introduced before the quark model was established, describes baryons within a meson field theory in the low energy domain of QCD.

The MIT bag model

As single quarks have never been observed, it is assumed that they only appear as bound states (mesons or baryons). One idea to model this behavior is to confine the quark wave function to a cavity. This cavity can be seen as a hadron and is surrounded by the QCD vacuum, cf. figure 2(a). In this idealized picture the vacuum is a perfect paramagnetic medium (with respect to color) because gluons couple to each other and can cause color magnetization. Therefore the color field is confined within the hadron, analogous to an electric field confined inside a cavity in a perfect conductor, cf. figure 2(b). Note that in this analogy the roles of $\mu$ and $\epsilon$ are interchanged.

So in a simple approach we can assume a sphere with radius $R$ that contains the quarks and solve the Dirac equation within that region. The Dirac equation for a free fermion with mass $m$ can be written as

$$i\gamma^\mu \partial_\mu \psi(x) = m\psi(x).$$

There are two solutions for the ground state corresponding to total angular momentum $j = \frac{1}{2}$ [6, 7]. We choose the solution for $\kappa = -1$

$$\psi_{-1}(x) = N \left( \sqrt{E + m} \frac{\mathbf{j}_0 \left( \frac{\epsilon x}{R} \right) }{E - m} U_m - \frac{\mathbf{j}_1 \left( \frac{\epsilon x}{R} \right) }{E - m} \right),$$

where $N$ is a normalization coefficient, $U_m$ a two-component spinor for angular momentum and $j_0$ and $j_1$ are spherical Bessel functions. The energy is given by

$$E = \sqrt{m^2 + \frac{x^2}{R^2}}$$

where $x$ is the quark momentum in units of $1/R$.

To confine the quarks inside the sphere we choose the boundary condition

$$n^\mu \overline{\psi} \gamma^\mu \psi = 0$$

so no current flow goes out of the sphere. Here $n^\mu$ is the outward normal to the sphere. This can be satisfied by

$$-i\gamma_\mu n^\mu \psi = \psi.$$

This boundary condition yields a constraint for the values of $x$ depending on the quark mass $m$ and the sphere radius $R$. We get

$$\frac{j_1(x)}{j_0(x)} = \sqrt{\frac{E + m}{E - m}} = \frac{\sqrt{(mR)^2 + x^2 + mR}}{\sqrt{(mR)^2 + x^2 - mR}}.$$  

The solution of this equation is shown in figure 3. In the ultrarelativistic limit ($m \to 0$) we obtain $x = 2.04$.

As one can see from equation (3) the model so far is unstable as the energy decreases with increasing $R$. Therefore the MIT bag model introduces a parameter $B$ that stabilizes the system. It can be interpreted as the vacuum pressure on the surface of the bag. The total energy becomes

$$E(R) = \sum_i N_i \sqrt{m_i^2 + \frac{x^2}{R^2}} + B \frac{4\pi R^3}{3}.$$
Figure 4: Hadron masses calculated with the bag model. All masses are quoted in GeV, $R_0$ in (GeV)$^{-1}$. The masses of the proton, $\Delta$ and $\omega$ were used to calculate the parameters. [9]

in the most general case with different quark masses $m_i$. In the limit $m \to 0$ we get

$$E(R) = \frac{2.04}{R} + B \frac{4\pi R^3}{3} .$$

(8)

By minimizing the energy we obtain an expression for the radius $R$ and the mass $M$ of the nucleus.

$$R = \left( \frac{2.04}{4\pi B} \right)^{1/4}, \quad M = \frac{4}{3} (4\pi B)^{1/4} (2.04 N)^{3/4} .$$

(9)

The parameter $B$ can be fitted using experimentally determined hadron masses.

With this simple model one can then calculate for example the masses of other hadrons or the spin structure of the nucleon [8]. With some corrections to the energy (quark and gluon interaction inside the bag for instance) the results are consistent with experimental data, cf. figure 4. The listed energy corrections are zero-point energy $E_0$, quark energy $E_Q$, color magnetic energy $E_M$ and color electric energy $E_E$. One problem is that the size of the bag is too large compared to the nucleon size if the parameters are fitted to match the mass. Another problem is that the model cannot provide a description for some processes in nucleon-nucleon scattering (e.g. pion exchange).

The Skyrme model

The basic idea of Skyrme was to introduce a field theory with mesons as gauge particles so baryons would interact with each other via the exchange of mesons. The wave functions that solve the field equations are not plain wave solutions but soliton waves, so called Skyrmions. The physical interpretation of these Skyrmions is still not completely resolved but according to latest ideas they are coherent states of baryons and excited baryons [10]. The remarkable thing about this model is that we can use it to describe hadrons and their interactions without taking their quark content into account.

Skyrme introduced the lagrangian

$$L = \frac{f_\pi^2}{4} \text{tr}(\partial_\mu U^\dagger \partial^\mu U) + \frac{1}{32g^2} \text{tr}[(U^\dagger \partial_\mu U)(U^\dagger \partial_\nu U)^2] .$$

(10)

with

$$U = \frac{\sigma + i \tau \cdot \pi}{f_\pi} ,$$

(11)

where $f_\pi$ is the decay constant for the pion, $g$ is the coupling constant $f_{\rho\pi\pi}$, $\tau$ is a vector containing the Pauli matrices and $\pi$ is the pion field. The first term in (10) was already known from the non-linear sigma model. Skyrme added the second term, which is of fourth order in the derivatives and realized that this would result in topological soliton solutions that minimize the energy. They have a property called topological charge which Skyrme identified as the baryon number. The fact that a soliton is not affected by dispersion can be seen as the conservation of baryon number, which is known to apply for strong interaction processes.

With this model it is also possible to calculate nucleon masses and other particle properties, cf. fig. 5. The accuracy is similar to that of the MIT bag model. Latest ideas about meson field theories claim that it is a valid approximation to QCD for low energies ($\sim$MeV).

### SUMMARY

As presented, both the MIT bag model and the Skyrme model are useful to calculate masses and other properties of hadrons. The Skyrme model describes only baryons though because the mesons are seen as the gauge particles of the meson field. The MIT bag model introduces many free parameters for energy corrections that maybe could be helpful to understand the physical processes inside the nucleus.