Identical particles

1. To simplify notation I only discuss models of particles in one dimensions. The generalization of what I write in these notes from one to three dimensions is straightforward: replace $x \rightarrow x$, $\int_\mathbb{R} dx \rightarrow \int d^3 x$, $-i\hbar \frac{\partial}{\partial x} \rightarrow -i\hbar \nabla$ etc.

2. A general model of $N$ particles in one dimension is given by the Hamiltonian

$$H_N = \sum_{j=1}^{N} \left( -\frac{\hbar^2}{2m_j} \frac{\partial^2}{\partial x_j^2} + V_j(x_j) \right) + \sum_{1 \leq j < k \leq N} v_{jk}(x_j, x_k)$$

(1)

with $m_j$ the particle masses, $V_j$ the external potential acting on particle $j$, and $v_{jk}(x, y)$ the two-body potential between particle $j$ and particle $k$. This acts on wave functions $\psi(x_1, x_2, \cdots, x_N)$. Note that the Hamiltonian in (1) is a sum of one-particle Hamiltonians

$$h_j = -\frac{\hbar^2}{2m_j} \frac{\partial^2}{\partial x_j^2} + V_j(x)$$

(2)

and two-body interaction terms $v_{jk}(x, y)$. Since all masses and potentials in (1) are different in general, this Hamiltonian describes *distinguishable particles*.

3. If $m_j = m$, $V_j = V$, and $v_{jk} = v$ independent of $j, k$ and such that $v(x, y) = v(y, x)$, i.e.

$$H_N = \sum_{j=1}^{N} \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_j^2} + V(x_j) \right) + \sum_{1 \leq j < k \leq N} v(x_j, x_k),$$

(3)

then we have a model of *non-distinguishable particles*. In this case the Hamiltonian commutes with the particle exchange operators defined as follows,

$$(T_{jk}\psi)(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_N) \equiv \psi(x_1, \ldots, x_k, \ldots, x_j, \ldots, x_N)$$

(4)

for all $j < k$.

Note that, if we allow for arbitrary wave functions $\psi, \tilde{\psi}$ and $T_{jk}\psi$ represent different states of the system, i.e., the particles still non-identical (even if they share the same features like mass etc.). Below we discuss what it means in quantum mechanics that the particles actually are identical.

4. \^2 More abstractly can a system of $N$ *non-distinguishable particles* be defined by a one-particle Hilbert space $\mathcal{H}$, a one-body Hamiltonian $h$ given by a self-adjoint operator on $\mathcal{H}$, and by a two-body interaction $v$ given by a self-adjoint operator

---

\(^1\)Version 1 (v1) by Edwin Langmann on Sept. 24, 2010; revised by EL on October 11, 2010

\(^2\)Paragraph marked with \^ (like this one) might be more difficult and can be skipped.
on $H \otimes H$ such that $Tv = v$ with the two-particle exchange operators defined as follows,
\[ T|f\rangle \otimes |g\rangle \equiv |g\rangle \otimes |f\rangle \]
for all one-particle states $|f\rangle$ and $|g\rangle$.
Then the Hilbert space for $N$ such particles is
\[ H^\otimes N \equiv H \otimes H \otimes \cdots \otimes H \ (N \text{ times}), \]
and the Hamiltonian describing $N$ such particles with this one-body and two-body terms is
\[ H_N = \sum_{j=1}^{N} h^{(j)} + \sum_{1 \leq j < k \leq N} v^{(jk)} \]
where the superscript indicates on which slot(s) of $H^\otimes N$ the operator acts, e.g.$^3$
\[ h^{(1)}|f_1\rangle \otimes |f_2\rangle \otimes \cdots \otimes |f_N\rangle \equiv h|f_1\rangle \otimes |f_2\rangle \otimes \cdots \otimes |f_N\rangle \]
\[ h^{(2)}|f_1\rangle \otimes |f_2\rangle \otimes \cdots \otimes |f_N\rangle \equiv |f_1\rangle \otimes h|f_2\rangle \otimes \cdots \otimes |f_N\rangle \]
\[ v^{(12)}|f_1\rangle \otimes |f_2\rangle \otimes \cdots \otimes |f_N\rangle \equiv \sum_{j,k,l,m} \langle j, k|v|l, m\rangle \langle l|f_1\rangle \otimes |k\rangle \langle m|f_2\rangle \otimes \cdots \otimes |f_N\rangle \]
\[ v^{(13)}|f_1\rangle \otimes |f_2\rangle \otimes \cdots \otimes |f_N\rangle \equiv \sum_{j,k,l,m} \langle j, k|v|l, m\rangle \langle l|f_1\rangle \otimes |f_2\rangle \otimes |k\rangle \langle m|f_3\rangle \otimes \cdots \otimes |f_N\rangle \]
etc. Note that the Hamiltonian in (7) commutes with all particle exchange operators $T_{jk} \equiv T^{(jk)}$ (these operators are defined similarly to $v^{(jk)}$ above).

**Remark:** We note in passing that the relation of the abstract formalism to our example is as follows: $\psi(x_1, x_2, \ldots, x_N) \equiv \langle x_1, x_2, \cdots, x_N|\psi\rangle$ and, in particular,
\[ \langle x_1, x_2, \cdots, x_N|f_1\rangle \otimes |f_2\rangle \otimes \cdots \otimes |f_N\rangle = f_1(x_1)f_2(x_2) \cdots f_N(x_N) \]
\[ (since \ |x_1, x_2, \cdots, x_N\rangle \equiv |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_N\rangle \text{ and } \langle x|f\rangle = f(x)). \]

5. Non-distinguishable particles in the quantum world are often identical, i.e., it is not possible to design any experiment where one can distinguish one particle from the other. To be specific, consider a two particle system with the Hamiltonian in (3) for $N = 2$ and a wave function $\psi(x_1, x_2)$. The physical interpretation of this wave function is as follows: \[ \text{"} |\psi(x_1, x_2)\rangle^2 \text{ is the probability density to find particle 1 at position } x_1 \text{ and particle 2 at position } x_2 \text{."} \] A wave function describing identical particles has the property that
\[ |\psi(x_1, x_2)\rangle^2 = |\psi(x_2, x_1)\rangle^2, \]
i.e., \[ \text{"} \text{the probability to measure that particle 1 is at position } x_1 \text{ and particle 2 at position } x_2 \text{ is the same as to measure that particle 1 is at position } x_2 \text{ and particle 2 at position } x_1 \text{."} \] This implies that
\[ (T\psi)(x_1, x_2) \equiv \psi(x_2, x_1) = e^{i\alpha} \psi(x_1, x_2). \]

$^3$While it is easy to write down $h^{(j)}$, the formulas for $v^{(jk)}$ are somewhat more awkward and require to expand $v$ in some complete orthonormal bases $|j\rangle$ in $H$: $v = \sum |j, k\rangle \langle j, k|v|l, m\rangle \langle l, m|$ with $|j, k\rangle = |j\rangle \otimes |k\rangle$. 
for some real $\alpha$, and since $T^2 = I$ we conclude $(e^{i\alpha})^2 = 1$, i.e. $e^{i\alpha} = +1$ and $-1$. This is a heuristic argument that makes plausible the following law of quantum physics.

6. **QM LAW V**: Identical particles in nature come in two kinds: bosons and fermions. Systems of identical bosons are described by wave functions that do not change under particle exchange. Systems of identical fermions are described by wave functions that change sign under particle exchange.

Thus in our previous example the allowed wave functions for bosons and fermions have the property that

\[(T\psi)(x_1, x_2) \equiv \psi(x_2, x_1) = \pm \psi(x_1, x_2) \quad (9)\]

with $+$ for bosons and $-$ for fermions. Note that the Hamiltonian in (3) for $N = 2$ commutes with the particle exchange operators, i.e., the Hamiltonian maps boson wave functions to boson wave functions, and similarly for fermions.

7. It is clear that the the space of all wave functions describing $N$ bosons is a subspace of the space of all $N$-particle wave functions, and similarly for fermions. How can one construct this subspace? For $N = 2$ this is easy: consider our example of wave functions $\psi(x_1, x_2)$. Then

\[\psi_\pm(x_1, x_2) = \frac{1}{2} \left( \psi(x_1, x_2) \pm \psi(x_2, x_1) \right) \quad (10)\]

obviously obeys $T\psi_\pm = \pm \psi_\pm$, i.e. it is a boson (for $+$) or fermion (for $-$) wave function. Note that we can write $\psi_\pm = A_\pm \psi$ with

\[A_\pm \equiv \frac{1}{2}(I \pm T) \quad (11)\]

a projection operator acting on the two-particle space: $A_\pm^2 = A_\pm$ just means that, if we apply $A_\pm$ to a wave function that is already a boson (for $+$) or fermion (for $-$) wave function, nothing happens. Moreover, $A_\pm$ is self-adjoint (check that!). We thus can naturally define the boson (for $+$) or fermion (for $-$) subspace of our two-particle Hilbert space $\mathcal{H}^\otimes 2$ as follows

\[\mathcal{H}^\otimes 2 \equiv A_\pm \mathcal{H}^\otimes 2. \quad (12)\]

While we motivated this construction using our simple example, all this naturally generalizes to any Hilbert space of two non-distinguishable particles (a possible generalization was discussed in Paragraph 1).

8. Let $f_n(x) = \langle x | f_n \rangle$, $n = 1, 2, \ldots$, be a complete orthonormal basis of 1-particle wave functions. Then $f_{n_1} f_{n_2}(x_1, x_2) \equiv f_{n_1}(x_1) f_{n_2}(x_2)$, $n_1, n_2 = 1, 2, \ldots$, is an orthonormal basis in the space of two-body wave functions, and

\[A_\pm (f_{n_1} f_{n_2})(x_1, x_2) = \frac{1}{2} \left( f_{n_1}(x_1) f_{n_2}(x_2) \pm f_{n_2}(x_1) f_{n_1}(x_2) \right) \quad (13)\]
is a basis in the two particle boson (for +) or fermion (for −) subspace. Note that,
to avoid double counting, one should restrict the quantum numbers to 1 ≤ n₁ ≤ n₂ < ∞ in the boson- and 1 ≤ n₁ < n₂ < ∞ in the fermion case.

It is important to note that the states in (13) are not normalized but have norm 1/2 (check that!). Thus a basis of normalized boson- and fermion states is

\[
\sqrt{2} \mathcal{A}_\pm(f_{n_1}, f_{n_2})(x_1, x_2) = \frac{1}{\sqrt{2}} (f_{n_1}(x_1)f_{n_2}(x_2) \pm f_{n_2}(x_1)f_{n_1}(x_2)).
\]  

(14)

More generally, if |fₙ⟩, n = 1, 2, ..., is an orthonormal basis in a one-particle Hilbert space \( \mathcal{H} \), then

\[
|f_1, f_2\rangle \equiv \sqrt{2} \mathcal{A}_\pm |f_{n_1}\rangle \otimes |f_{n_2}\rangle = \frac{1}{\sqrt{2}} (|f_{n_1}\rangle \otimes |f_{n_2}\rangle \pm |f_{n_2}\rangle \otimes |f_{n_1}\rangle)
\]  

(15)

is a basis of normalized states in \( \mathcal{H}^{\otimes 2}_\pm \) for 1 ≤ n₁ ≤ n₂ < ∞ in the boson case (+) and 1 ≤ n₁ < n₂ < ∞ in the fermion case (−).

9. Further below we present a more powerful formalism to construct boson and fermion wave functions based on creation- and annihilation operators. To understand and appreciate this formalism one should know the more down-to-earth (but also more tedious) construction which we present below. However, it you are ready to accept the powerful formalism you may want to skip the reminder of this section (i.e. all paragraphs marked by a * below).

10. * Can we generalize this to \( N \)-particle spaces? Yes: one can construct self-adjoint projection operators \( \mathcal{A}_\pm \) on \( \mathcal{H}^{\otimes N} \) such that

\[
\mathcal{H}^{\otimes N}_\pm \equiv \mathcal{A}_\pm \mathcal{H}^{\otimes N}
\]  

(16)

is the boson (for +) or fermion (for −) subspace of \( \mathcal{H}^{\otimes N} \).

11. * By QM law V, a \( N \)-particle wave function \( \psi(x_1, x_2, \ldots, x_N) \) describes boson (for +) or fermions (for −) and only if

\[
(T_{jk}\psi)(x_1, x_2, \ldots, x_N) = \pm \psi(x_1, x_2, \ldots, x_N)
\]  

(17)

for all \( j < k \). The operators \( T_{jk} \) exchanges particle \( j \) with particle \( k \).

More generally, we can permute particles: recall that a \textbf{permutation} \( P \) of \( N \) objects is an invertible map\(^4\) \( \{1, 2, \ldots, N\} \rightarrow \{1, 2, \ldots, N\} \), e.g. \( P(1, 2, 3) \equiv (P1, P2, P3) = (3, 2, 1) \) or \( P(1, 2, 3) = (2, 1, 3) \) are two examples of permutations of 3 objects. In the following specify a permutation \( P \) by writing down \( (P1, P2, \ldots, PN) \) (i.e. we write \( (2, 1, 3) \) short for the permutation \( P \) such that \( P(1, 2, 3) = (2, 1, 3) \) etc.\)

\(^4\)We denote as \( \{1, 2, \ldots, N\} \) the set containing the numbers 1, 2, ..., \( N \) (the order is irrelevant), and as \( (P1, P2, \ldots, PN) \) an ordered set (the order matters).
One can show that every permutation can be written as a sequence of particle exchanges, i.e.

\[ P = T_{j_1 k_1} T_{j_2 k_2} \cdots T_{j_p k_p}, \tag{18} \]

and a permutation is called even if \( p \) is even and odd if \( p \) is odd. One defines \((\pm 1)^P \equiv (\pm)^p\), i.e. \((+1)^P = 1\) (always) and \((-1)^P = -1\) for odd permutations \( P \) and \((-1)^P = 1\) for all permutations \( P \). We also define \((+1)^P = 1\) for all permutations \( P \). We also recall that the set of all different permutations of \( N \) objects is called \( S_N \). This set is a group, and it contains \( N! \) elements.

For example, \( S_3 \) contains the six permutations \((1, 2, 3), (2, 3, 1), (3, 1, 2), (2, 1, 3), (3, 2, 1), (1, 3, 2)\), where the first three permutations are even and the last three odd.

One can show that the generalization of the operator in (11) to \( N \) particles is

\[ (A_\pm \psi)(x_1, x_2, \ldots, x_N) = \frac{1}{N!} \sum_{P \in S_N} (-1)^p (P \psi)(x_1, x_2, \ldots, x_N), \tag{19} \]

i.e. the wave functions \( \psi_\pm \equiv A_\pm \psi \) obeys the conditions in (17).

12. The generalization of this to arbitrary tensor products \( \mathcal{H} \otimes^N \) is as follows,

\[ A_\pm |f_1\rangle \otimes |f_2\rangle \otimes \cdots \otimes |f_N\rangle \equiv \frac{1}{N!} \sum_{P \in S_N} (-1)^P |f_{P1}\rangle \otimes |f_{P2}\rangle \otimes \cdots \otimes |f_{PN}\rangle. \tag{20} \]

The operator \( A_\pm \) thus defined is self-adjoint and such that \( A_\pm = A_\pm^2 \). (You are encouraged to check all these statements — to do that you will need some facts about the permutation group \( S_N \)). One can show that this definition implies that wave functions

\[ |f_1, f_2, \ldots, f_N\rangle_\pm \equiv \sqrt{N!} A_\pm |f_1\rangle \otimes |f_2\rangle \otimes \cdots \otimes |f_N\rangle \]

have the following scalar product

\[ \langle f_1, f_2, \ldots, f_N | g_1, g_2, \ldots, g_N \rangle = \sum_{P \in S_N} (-1)^P \langle f_1, g_{P1} | f_2, g_{P2} \rangle \cdots \langle f_N, g_{PN} \rangle. \tag{22} \]

Note that, in the fermion case \((-\rangle\) and for \( N = M \), the r.h.s. of this is equal to the determinant of the matrix \( \langle f_j | g_k \rangle_{j,k=1}^N \). This implies that, if \( |f_n\rangle \), \( n = 1, 2, \ldots \), is a basis of orthonormal states in the one-particle Hilbert space \( \mathcal{H} \), then \( |f_{n_1}, f_{n_2}, \ldots, f_{n_N}\rangle_\pm \) for

\[ 1 \leq n_1 \leq n_2 \leq \cdots \leq n_N < \infty \text{ in the boson case } (+) \]
\[ 1 \leq n_1 < n_2 < \cdots < n_N < \infty \text{ in the fermion case } (\mp) \tag{23} \]

is a complete basis of orthonormal states in \( \mathcal{H}_\pm \otimes^N \).

---

\(^5\)Note that the following representation of a permutation is not unique, but \((-)^p\) is the same for each such representation, and thus “even/odd” below is well-defined.
Boson- and fermion Fock spaces

13. The following is an introduction to problems on many-body systems of bosons and fermions. As discussed in the lectures, one can construct boson- and fermion states by symmetrizing and anti-symmetrizing general many-particle states, respectively. This is conceptually simple but rather tedious for practical computations. A much more convenient method is to use boson- or fermion creation- and annihilation operators. In the following I describe how they are defined, and how they are used to construct Fock spaces. To digest these notions I recommend that you study the suggested exercises.

14. The Fock space is a Hilbert space containing many-particle states with an arbitrary number of particles, \( N = 0, 1, 2, \ldots \). This is convenient since this allows to have operators that change the particle number: creation operators that add one particle to a given state, and annihilation operators that remove a particle from a given states. It then is possible to construct a basis of states starting with the zero-particle state \( | \Omega \rangle \) and applying creation operators to it.

15. We use the following notation: For bosons we use commutators defined for a pair of operators \( A, B \) as follows,

\[
[A, B]_- \equiv [A, B] = AB - BA,
\]

and for fermions we use anti-commutators

\[
[A, B]_+ \equiv \{A, B\} = AB + BA.
\]

The notation \([A, B]_\pm\) is convenient to write down formulas that are true for bosons, but otherwise one uses \([A, B]\) and \(\{A, B\}\).

16. Consider a many-body system of bosons where one particle is described by a Hilbert space \( \mathcal{H} \) with a complete orthonormal basis \( | j \rangle, j = 1, 2, \ldots, n \) with \( n \leq \infty \), i.e.

\[
\sum_j |j\rangle \langle j| = I, \quad \langle j|k\rangle = \delta_{jk}.
\]

(In our discussion above \( \mathcal{H} \) was often infinite dimensional \( n = \infty \), but in many applications \( n \) is finite.) Assign to each ket \( |j\rangle \) a so-called creation operators \( a_j^\dagger \) and to each bra \( \langle j| \) a so-called annihilation operators \( a_j \) that are defined by the following (anti-) commutator relations,

\[
[a_j, a_k^\dagger]_\pm = \delta_{j,k}
\]

\[
[a_j, a_k]_\pm = [a_j^\dagger, a_k^\dagger]_\pm = 0
\]

for all \( j, k = 1, 2, \ldots, n \). Introduce also a special state \( | \Omega \rangle \) called the vacuum or zero-particle state defined by the following relations,

\[
a_j| \Omega \rangle = 0 \quad \forall j, \quad \langle \Omega | \Omega \rangle = 1.
\]

Note that, in the boson case, these are just the relations of \( n \) decoupled harmonic oscillators.
17. We can consider states

\[ |j_1, j_2, \cdots, j_N \rangle \equiv a_{j_1}^\dagger a_{j_2}^\dagger \cdots a_{j_N}^\dagger |\Omega \rangle. \tag{29} \]

The boson Fock space \( \mathcal{F}_B(\mathcal{H}) \) is the Hilbert space of all possible linear combinations of the states in (29). It is important that the inner product of the states in (29) is fully determined by the relations in (27), (28), and by the rule that \( a_j \) is the Hilbert space adjoint of \( a_{j}^\dagger \), i.e.

\[ (a_j)^\dagger = a_j, \quad (a_j)^\dagger = a_j. \tag{30} \]

Indeed, one can prove that the scalar product

\[ \langle k_1, k_2, \cdots, k_M | j_1, j_2, \cdots, j_N \rangle \equiv (|k_1, k_2, \cdots, k_M \rangle)^\dagger |j_1, j_2, \cdots, j_N \rangle \tag{31} \]

is non-zero only if \( N = M \), and if this is the case it is equal to

\[ \begin{cases} \pm 1^P & \text{if } (k_1, k_2, \cdots, k_N) \text{ is a permutation } P \text{ of } (j_1, j_2, \cdots, j_N) \\ 0 & \text{otherwise} \end{cases} \tag{32} \]

with \( (+1)^P = 1 \) always (boson case) and \( (-1)^P = +1/-1 \) for even/odd permutations (fermion case).

18. It is natural to assign to each one-particle ket- and bra, \( |f \rangle = \sum_j f_j |j \rangle \) and \( \langle f | = \sum_j \overline{f}_j \langle j | \), creation- and annihilation operators \( a^\dagger(f) \) and \( a(f) \) as follows,

\[ a^\dagger(f) = \sum_j f_j a_{j}^\dagger, \quad a(f) = \sum_j \overline{f}_j a_j. \tag{33} \]

These satisfy the following relations generalizing the ones in (27),

\[ [a(f), a^\dagger(g)]_\pm = \langle f | g \rangle, \quad [a(f), a(g)]_\mp = 0 \tag{34} \]

e etc. for all one-particle states \( |f \rangle \) and \( |g \rangle \). The physical interpretation of these operators is as follows: if \( |\eta \rangle \) is any state with \( N \) particles, then \( a^\dagger(f)|\eta \rangle \) is a state with \( N + 1 \) particles and the additional particles is in the one-particle state \( |f \rangle \), and \( a(f)|\eta \rangle \) is a state with \( N - 1 \) particles where a particle in the one-particle states \( |f \rangle \) is removed (if no such particle is present in \( |\eta \rangle \) then one gets zero). This is the reason for the name of these operators.

19. Note that, in the fermion case,

\[ a^\dagger(f)^2 = \frac{1}{2} [a^\dagger(f), a^\dagger(f)]_+ = 0. \tag{35} \]

This corresponds to the Pauli principle: it is not possible to have any many-fermion state where two electrons are in the same one-particle state. There is no such restriction for bosons.
20. Using creation operators one can obtain many-body states by applying several creation operators to the vacuum:

\[ |f_1, f_2, \ldots, f_N \rangle_\pm \equiv a_1^\dagger(f_1)a_2^\dagger(f_2) \cdots a_N^\dagger(f_N)|\Omega \rangle \] (36)

with one-particles states \( |f_j \rangle \). This state corresponds to having \( N \) particles in the one-particles states \( |f_j \rangle, j = 1, 2, \ldots, N \) (but, since the particles are identical, one cannot tell which particles is in which one-particle state). It is important to note that not all \( N \)-particle states are of these form: states as in (36) are called **uncorrelated**, and general **correlated** states can only be written as linear combinations of such states. However, to get a basis not all such states need to be taken: it is enough to restrict to states as in (29), for example.

21. To show that the present formalism is equivalent to the one using symmetrized tensor product wave function one can show that the states in (36) are in one-to-one correspondence with the states in (21) and, in particular, that they have exactly the scalar products in (22) (check that!).

22. Many models of interest can be defined in a simple manner using the formalism above: typically one is interested in models with one- and two body interactions. A one-body Hamiltonian is defined by a self-adjoint operator \( h \) on the one-body Hilbert space \( \mathcal{H} \), and a two-body Hamiltonian by a self-adjoint operator \( v \) on the corresponding two-particle Hilbert space. They can be written as

\[ h = \sum_{j,k} h_{jk} |j\rangle \langle k| \text{ and } v = \sum_{j,k,l,m} v_{jklm} |j,k\rangle \langle l,m| \] (37)

with \( |j,k\rangle \equiv |j\rangle \otimes |k\rangle \) and the matrix elements \( h_{jk} = \langle j|h|k \rangle \) and \( v_{jklm} = \langle j,k|v|l,m \rangle \). Self-adjointness and the assumed boson fermion symmetry imply \( h_{jk} = \overline{h_{kj}}, v_{jklm} = \overline{v_{lmjk}}, \) and

\[ v_{jklm} = \pm v_{kjlm} = \pm v_{jkm} \] (38)

The Hamiltonian describing an arbitrary number of identical bosons is an operator on the Fock space \( \mathcal{F}_p \) given by

\[ H = H_0 + H_{\text{int}}, \quad H_0 = \sum_{jk} h_{jk} a_j^\dagger a_k, \quad H_{\text{int}} = \frac{1}{2} \sum_{jklm} v_{lujm} a_j^\dagger a_k^\dagger a_l a_m. \] (39)

Many models of interest in condensed matter- and nuclear physics are of this type.

23. This formalism is useful since, for models without two-body interactions (i.e. if \( H = H_0 \)), the problem to find eigenstates of the many-body Hamiltonian can be reduced to finding the eigenstates of the one-particle Hamiltonian \( h \). To be more specific: assume that \( |f_j \rangle = \sum_k (f_j)_k |k \rangle \) are orthonormal eigenstates of the one-particle Hamiltonian \( h \) with corresponding eigenvalues \( E_j \). Then \( \tilde{a}_j^{(1)} \equiv a^{(1)}(f_j) \) obey the same (anti-) commutator relations as the operators \( a_j \). Moreover, these operators allow us to write

\[ H_0 = \sum_j E_j \tilde{a}_j^{\dagger} \tilde{a}_j. \] (40)
Consider now the states

$$|N_1, N_2, \ldots, N_n⟩ = \frac{(\hat{a}_1^\dagger)^{N_1}}{\sqrt{N_1!}} \frac{(\hat{a}_2^\dagger)^{N_2}}{\sqrt{N_2!}} \cdots \frac{(\hat{a}_n^\dagger)^{N_n}}{\sqrt{N_n!}} |Ω⟩$$

(41)

with $N_j = 0, 1, 2, \ldots$ in the boson case and $N_j = 0, 1$ in the fermion case (we set $0! = 1$). One can show that these states are orthonormal, complete, and common eigenstates of the operators $\hat{a}_j^\dagger \hat{a}_j$ with corresponding eigenvalue $N_j$. Thus the states in (41) are eigenstates of the Hamiltonian $H_0$ with corresponding eigenvalue

$$\mathcal{E} = \sum_j E_j N_j.$$  

(42)

With that it is possible to compute many physical properties of such a model of non-interacting bosons analytically. One can understand quite a lot of interesting physics by analyzing such solutions (you will see various examples in advanced courses on condensed matter physics or nuclear physics).

24. In case there are interactions the problem to find eigenstates and eigenvalues of a Hamiltonian as in (39) is very difficult and can be, in general, only solved in some approximation. One method that is often remarkably successful is Hartree-Fock theory: Use states $|η⟩ \equiv |f_1, f_2, \ldots, f_N⟩_±$ as in (36) as variational states, i.e. compute $⟨η|H|η⟩$, and choose the orthonormal functions $f_j$ such that this expectation value is minimized. Computing this expectation value and minimizing it leads to equations that can be often solved with reasonably small numerical effort (the latter is usually done with a computer).

25. In many cases of interest the 1-particle space is in position representation. As an example I consider the many-body Hamiltonian in (3) describing identical particles with mass $m$ moving on the real line and interacting with an external potential $V$ and two-body interactions $v$. In this case, one-particle eigenfunctions are $|x⟩$ with $x ∈ \mathbb{R}$, one usually writes $\hat{ψ}^\dagger(x)$ rather than $a_x^\dagger$, i.e., the relations in (27) etc. are written as follows,

$$[ψ(x), ψ^\dagger(y)]_± = δ(x − y)$$

$$[ψ(x), ψ(y)]_± = [ψ^\dagger(x), ψ^\dagger(y)]_± = 0$$

$$ψ(x)|Ω⟩ = 0$$

(43)

for all $x, y ∈ \mathbb{R}$. Using this the many-body Hamiltonian in (3) can be written as

$$H = H_0 + H_{int}, \quad H_0 = \int dx \hat{ψ}^\dagger(x) \left( -\frac{ℏ^2}{2m} \frac{∂^2}{∂x^2} + V(x) \right) \hat{ψ}(x)$$

$$H_{int} = \frac{1}{2} \int dx \int dy \hat{ψ}^\dagger(x) \hat{ψ}^\dagger(y) v(x − y) \hat{ψ}(y) \hat{ψ}(x)$$

(44)

where all integrations are over $\mathbb{R}$. The $\hat{ψ}^{(f)}(x)$ are often called boson field operators.

---

6There are a few exactly solvable models with interactions, but their solution requires special techniques.
26. Note that, in the non-interacting case (i.e. if $H = H_0$), the Hamiltonian in (44) can be diagonalized as follows: assume that $f_j(x)$ are orthonormal eigenfunctions of the one-particle Hamiltonian $h = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$ with corresponding eigenvalues $E_j$. Define

$$a_j = \int dx f_j(x) \hat{\psi}(x), \quad a_j^\dagger = \int dx \hat{\psi}^\dagger(x) f_j(x)$$

(45)

equivalent to $\hat{\psi}(x) = \sum_j a_j f_j(x)$ etc. Observe that these operators $a_j^{(\dagger)}$ obey the relations in (27). Moreover,

$$H_0 = \sum_j E_j a_j^\dagger a_j.$$  

(46)

It thus is easy to construct all eigenstates and eigenvalues of $H_0$ once ones know the eigenstates and eigenvalues of the one-particle Hamiltonian.

27. Note that, in the non-interacting case (i.e. if $v = 0$), the integral on the r.h.s. in (44) looks exactly like the formula for the expectation value of the one-particle Hamiltonian in the one-particle states $\psi(x)$:

$$\langle \psi | \hat{h} | \psi \rangle = \int dx \overline{\psi(x)} \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \psi(x).$$

This is one reason for why the Fock space formalism described above often is called 2nd quantization: 1st quantization amounts to going from classical- to quantum mechanics, and it formally amounts to replacing a classical position- and momentum variable $x$ and $p$ by operators,

$$x \rightarrow \hat{X}, \quad p \rightarrow \hat{P}$$

such that $[\hat{P}, \hat{X}] = -i\hbar$. In a similar manner, going from a 1- to a many-body model formally amounts to replacing 1-particle wave functions by operators

$$\psi(x) \rightarrow \hat{\psi}(x) \quad \overline{\psi(x)} \rightarrow \hat{\psi}^\dagger(x)$$

with the relations in (43). I emphasize that the name “2nd quantization” is somewhat misleading: as I tried to explain, it is nothing but a concise summary of a convenient mathematical framework to represent quantum systems with an arbitrary number of identical bosons or fermions.

28. Field operators and Fock spaces also play an important role in quantum field theory. One important difference to many-body systems described here is that then one also has Hamiltonians that do not conserve the particle number, i.e. one has terms like $\sum_{jk} \Delta_{jk} a_j^\dagger a_k^\dagger + h.c.$ (“h.c.” is short for “hermitean conjugate” and means that one should add the terms obtained by taking the hermitean conjugate of the terms written). It thus no longer is possible to restrict the model to subspace with a fixed particle number. This makes things more difficult and more interesting.