1. Consider an Ising chain with nearest-neighbour interactions,

\[ H = -J_1 \sum_{i=1}^{N-1} S_i S_{i+1}, \]

where the spins take values +1 and -1. Use mean-field theory to derive a self-consistent equation for the magnetization. Determine the critical temperature \( T_c \) and the critical exponent \( \beta \) (see below for a definition) from this self-consistency equation.

2. A model closely related to the Ising model is the Potts model. Consider the one-dimensional Potts model,

\[ H = -J \sum_{i=1}^{N-1} \delta_{S_i S_{i+1}}, \]

where the “spins” \( S_i \) take the values 0 and 1, and \( \delta \) is the Kronecker delta, \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) if \( i \neq j \). Calculate the partition function \( Z \) exactly, and determine the thermal expectation value of the Hamiltonian, \( \langle H \rangle = \text{Tr}(H e^{-\beta H})/Z \), in the limit of a very long chain. You can use either open or periodic boundary conditions in your calculations.

3. The Landau free energy that describes the tricritical point is of the form

\[ G(T, m) = \frac{1}{2} b m^2 + \frac{1}{6} f m^6 - mh, \]

with \( b = b_0 (T - T_c) \) and \( f, b_0 > 0 \). The variable \( h \) denotes a magnetic field. Determine the critical exponents \( \alpha, \beta, \gamma \) and \( \delta \). Check if Rushbrooke’s law, \( \alpha + 2\beta + \gamma = 2 \), holds.

4. Consider a system with only one relevant variable, which we take to be the reduced temperature \( t = (T - T_c)/T_c \). Use arguments from renormalization group theory to motivate the scaling form of the free energy,

\[ g(t) \sim t^{-d} g(t^*). \]
Show how the exponent $y_h$ can be determined in terms of the the recursion relation for the temperature, $T' = R(T)$, close to the fixed point. Use the more general scaling form

$$g(t, h) \sim t^{-d} g(t^{y_t}, t^{y_h} h)$$

to determine the critical exponent $\gamma$ in terms of the exponents $y_t$ and $y_h$.

5. Experiments are performed to measure the susceptibility of a material that is well described by a one-dimensional Ising model,

$$H = -J \sum_{i=1}^{N-1} S_i S_{i+1} - h \sum_{i=1}^{N} S_i,$$

In the experiment magnetic ions are randomly substituted by non-magnetic ions. In the Ising model this corresponds to removing spins with a probability $1 - p$. The measurements are performed at very low temperatures $T << J$. A set of samples with varying dilution strength $p$ is prepared, and the susceptibility is measured as $p$ approaches the percolation limit, $p_c$. The experiments are found to be well described by the following relation,

$$\chi \sim (p - p_c)^{-\gamma}.$$

Your job as a theorist is to calculate the exponent $\gamma$. Show all the steps of your calculation.

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Definitions of the critical exponents:

$$c(t, h = 0) = -T\frac{\partial f}{\partial T^{2}} \sim |t|^{-\alpha}$$

$$m(t, h = 0) = \frac{\partial f}{\partial h}\bigg|_{h=0} \sim (-t)^{\beta}$$

$$\chi(t, h = 0) = \frac{\partial m}{\partial h} \sim |t|^{-\gamma}$$

$$m(t = 0, h) \sim |h|^{\frac{1}{c}}$$

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*LYCKA TILL! / GOOD LUCK!*
TENTAMEN I STATISTISTISK MEKANIK

Statistisk mekanik 5A1350 för F3
Onsdag 2004-05-26, kl. 08.00-13.00

SOLUTIONS

1. Assume that spin $S_i$ sees the average magnetization of the other spins, $S_j \rightarrow \langle S_j \rangle = m$. The mean-field Hamiltonian can then be written as

$$H^{MF} = -J_1 \sum_i S_i \sum_m m.$$ 

On a chain there are two nearest neighbours (nn), and therefore

$$H^{MF} = -2m J_1 \sum_i S_i.$$ 

This is a non-interacting Hamiltonian, and we can consider the mean-field Hamiltonian for a single spin,

$$H_i^{MF} = -2m J_1 S_i.$$ 

The self-consistency equation for the magnetization is

$$m = \frac{\sum_{S_i=\pm 1} S_i e^{-\beta H_i^{MF}}}{\sum_{S_i=\pm 1} S_i e^{-\beta H_i^{MF}}} = \tanh[2J_1\beta m].$$ 

![Graph of tanh function](image)

The critical temperature is determined by the derivative of $m$ and $\tanh[2J_1\beta m]$ being equal at the origin,

$$\frac{d}{dm} (m = \tanh[2J_1\beta m]) \big|_{m=0},$$

3
which leads to the equation $1 = 2J_1 \beta_c$, and finally $1/\beta_c = k_b T_c = 2J_1$.

In order to determine the critical exponent $\beta$ we Taylor expand the self-consistency equation for small values of $m$,

$$m = \tanh[2J_1 \beta m] = \tanh\left[\frac{T_c}{T} m\right] \approx \frac{T_c}{T} m - \frac{1}{3} \frac{T_c^3}{T} m^3.$$ 

Solving for $m$ we obtain $m^2 = 3 \frac{T_c^2}{T} \frac{T_c - T}{T_c}$, and close to $T_c$ we see that $m \sim (T_c - T)^{\frac{1}{2}}$, and therefore $\beta = \frac{1}{2}$.

2. Consider open boundary conditions,

$$Z_n = \sum_{S_1, \ldots, S_N} e^{\beta J \sum_{i=1}^{N-1} \delta S_i S_{i+1}}$$

$$= \sum_{S_1, \ldots, S_{N-1}} e^{\beta J \sum_{i=1}^{N-2} \delta S_i S_{i+1}} \sum_{S_N} e^{\beta J \delta S_N S_{N-1} S_N}$$

$$= \sum_{S_1, \ldots, S_{N-1}} e^{\beta J \sum_{i=1}^{N-2} \delta S_i S_{i+1} (1 + e^{\beta J})}$$

$$= \sum_{S_1, S_2} e^{\beta J \delta S_1 S_2} (1 + e^{\beta J})^{N-2}$$

$$= 2(1 + e^{\beta J})^{N-1}$$

We have

$$\langle H \rangle = \frac{\text{Tr}(He^{-\beta H})}{\text{Tr}e^{-\beta H}} = -\frac{\partial \ln Z}{\partial \beta},$$

and since $\ln Z = \ln 2 + (N - 1) \ln(1 + e^{\beta J})$ we get $\lim_{N \to \infty} \ln Z = N \ln(1 + e^{\beta J})$.

Therefore

$$\langle H \rangle = -\frac{\partial N \ln(1 + e^{\beta J})}{\partial \beta} = -N \frac{Je^{\beta J}}{1 + e^{\beta J}} = \frac{-NJ}{1 + e^{\beta J}}$$

We can check that $\lim_{\beta \to \infty} \langle H \rangle = -NJ$, which corresponds to all spins in the same state (0 or 1).

3. The magnetization $\tilde{m}$ is given by

$$\frac{\partial G}{\partial \tilde{m}} = 0 = b \tilde{m} + f \tilde{m}^5 - h = 0$$

Set $h = 0$ and we get

$$\tilde{m} = \left(\frac{-b}{f}\right)^{\frac{1}{4}} = \left(\frac{-b_0 (T - T_c)}{f}\right)^{\frac{1}{4}}$$

and $\tilde{m} \sim (T_c - T)^{\frac{1}{4}}$ (for $T < T_c$) with $\beta = \frac{1}{4}$.
Determine the susceptibility \( \chi = \frac{\partial m}{\partial h} \)

\[
\frac{\partial}{\partial h}(b\tilde{m} + f\tilde{m}^5 - h) = b\chi + 5f\tilde{m}^4\chi - 1 = 0
\]

and

\[
\chi = \frac{1}{b + 5f\tilde{m}^4}
\]

Considering \( T > T_c \) we get \( \chi = 1/b = 1/b_0(T - T_c)^{-1} \) and \( \chi \sim (T - T_c)^{-\gamma} \) with \( \gamma = 1 \).

The equation of state at \( T = T_c \) is given by \( f\tilde{m}^5 - h = 0 \) and it follows that

\[
\tilde{m} = \left(\frac{h}{f}\right)^\frac{1}{5}.
\]

Therefore \( \tilde{m}(T = T_c) \sim h^\frac{1}{5} \), with \( \delta = 5 \).

Finally the specific heat \( C = T\frac{\partial^2 G}{\partial T^2} \). As \( T \to T_c^- \) we get

\[
G = \frac{1}{2}b_0(T - T_c)\frac{b_0^5(T_c - T)^\frac{1}{5}}{f^\frac{5}{6}} + \frac{1}{6}b_0^5(T_c - T)\frac{b_0^\frac{5}{6}}{f^\frac{5}{6}} \propto (T_c - T)^\frac{5}{6}
\]

and therefore \( C = T\frac{\partial^2 G}{\partial T^2} \propto T(T_c - T)^{-\frac{1}{5}} \) and \( C \propto (T_c - T)^{-\alpha} \), with \( \alpha = \frac{1}{2} \).

We have \( \alpha + 2\beta + \gamma = \frac{1}{5} + 2\frac{1}{5} + 1 = 2 \) and Rushbrooke’s law holds.

4. Assuming that the transformed Hamiltonian has the same functional form as the original Hamiltonian it follows that the free energy \( G \) will also be of the same form. Consider the transformation of the free energy per spin, \( N g(t) = N' g(t') \). If the factor of rescaling is \( l \), then the number of spins transforms as \( N' = l^{-d} N \). The temperature \( T \) transforms as \( T' = R(T) \). Linearizing close to the fixed point \( T_c \) we get

\[
T' = R_l(T) = R_l(T_c + \delta T) = T_c + \frac{\partial R_l}{\partial T}|_{T=T_c}\delta T,
\]

and \( \delta T' = \frac{\partial R_l}{\partial T}|_{T=T_c}\delta T \), and so \( t' = \frac{\partial R_l}{\partial T}|_{T=T_c} t = \lambda_l t \). Rescaling twice with length scale \( l \) should yield the same result as rescaling once with length scale \( l^2 \), and therefore \( \lambda_l = \lambda_l \lambda_l \), leading to \( \lambda_l = t^{y_l} \). Putting it all together we get

\[
g(t) = l^{-d}g(t^{y_l}),
\]

with

\[
y_l = \frac{\ln(\frac{\partial R_l}{\partial T}|_{T=T_c})}{\ln l}.
\]

To calculate \( \gamma \) we use \( \chi(t, h = 0) = \frac{\partial m}{\partial h} \sim |t|^{-\gamma} \). First we determine the magnetization

\[
\frac{\partial}{\partial h}(g(t, h) = l^{-d}g(l^{y_h}t, l^{y_h}h))
\]

\[
m(t, h) = l^{-d+y_h}m(l^{y_h}t, l^{y_h}h).
\]
From here we calculate the susceptibility,

\[
\frac{\partial}{\partial h}(m(t, h)) = t^{-d-2y_h}m(p^n t, l^n h) \quad \chi(t, h) = t^{-d-2y_h} \chi(p^n t, l^n h).
\]

Choosing \( h = 0 \) and \( l = |t|^{-\frac{1}{y_t}} \) we get

\[
\chi(t, h) = t^{-\frac{2y_h - d}{y_t}} \chi(\pm 1, 0) \sim t^{-\gamma},
\]

and \( \gamma = \frac{2y_h - d}{y_t} \).

5. All spins within each cluster will be aligned. Consider a cluster of size \( s \). The magnetization is

\[
M_{\text{cluster}} = s \left( \frac{e^{\frac{sh}{kT}} - e^{-\frac{sh}{kT}}}{e^{\frac{sh}{kT}} + e^{-\frac{sh}{kT}}} \right) = s \tanh\left( \frac{sh}{kT} \right),
\]

and the magnetization per spin is (below the percolation threshold \( p_c = 1 \))

\[
m = \sum_{s=1}^{\infty} s n_s \tanh\left( \frac{sh}{kT} \right).
\]

The susceptibility is given by \( (\tanh(x) \sim x \text{ for small } x) \)

\[
\chi = \sum_{s=1}^{\infty} s^2 n_s \frac{d}{kT}.
\]

For the one-dimensional percolation problem the normalized cluster number \( n_s = p^n(1-p)^2 \), and

\[
kT\chi = (1-p)^2 \sum_{s=1}^{\infty} s^2 p^n
\]

\[
= (1-p)^2 \left( p \frac{d}{dp} \right)^2 \sum_{s=1}^{\infty} p^n
\]

\[
= (1-p)^2 \left( p \frac{d}{dp} \right)^2 \frac{p}{1-p}
\]

\[
= (1-p)^2 \frac{p(1+p)}{(1-p)^3}
\]

\[
= p(1+p)
\]

The percolation threshold is \( p_c = 1 \) and

\[
\chi \propto (p_c - p)^{-\gamma},
\]

with \( \gamma = 1 \).