1. Consider a linear Ising chain with nearest and next nearest interactions:
\[ H = -J_1 \sum_i S_i S_{i+1} - J_2 \sum_i S_i S_{i+2}, \]
where the spins may be \( \pm 1 \) while \( J_1 \) and \( J_2 \) are positive constants. Derive a self-consistent mean-field equation for the magnetisation. Determine the phase transition temperature and the critical exponent for the magnetisation (\( \beta \)).

2. In a Landau theory the free energy may be written as
\[ f(T, m) = a \frac{1}{2} (T - T_c)m^2 + c \frac{1}{6} m^6, \]
with \( a \) and \( c \) being positive constants. Determine the specific heat exponent \( \alpha \).
5. The Ginzburg-Landau model for the free energy reads in its simplest form in three dimensions

\[ F(m, T) = \int d^3r \left( \frac{b_0(T - T_c)}{2} [m(r)]^2 + \frac{c}{2} [m(r)]^4 + \frac{f}{2} [\nabla m(r)]^2 \right), \]

with \(b_0, c\) and \(f\) being positive constants. Apply a local perturbation, \(h_0\delta(r)\), to the uniform equilibrium magnetisation and show that the response to the perturbation will be \(h_0e^{-r/\xi(T)}/r\)! Determine the correlation length \(\xi(T)\) close below and above the phase transition!

Definitions of some critical exponents:

\[
    \begin{align*}
    c(t, h = 0) &= -T \frac{\partial^2 f}{\partial T^2} \propto |t|^{-\alpha} \\
    m(t, h = 0) &= -\frac{\partial f}{\partial h} \propto (t)^\beta \\
    \chi(t, h = 0) &= \frac{\partial m}{\partial h} \propto |t|^{-\gamma} \\
    m(t = 0, h) &= -\frac{\partial f}{\partial h} \propto |h|^{1/\delta} \text{ sign}(h)
    \end{align*}
\]

Solutions

1. Assuming that a spin \(S_i\) sees \(\langle S_j \rangle = m\) at another site \(j\), we obtain:

\[ H_{MF} = -2mJ_1 \sum_i S_i - 2mJ_2 \sum_i S_i = 2mJ \sum_i S_i, \]

with \(J = J_1 + J_2\). We have used that each spin has two nearest and two next nearest neighbours in the linear chain. This decouples the spins and we can derive as self consistent equation for the magnetisation, \(m\)

\[
m = \langle S_i \rangle = \frac{2mJe^{2mJ/k_BT} - 2mJe^{-2mJ/k_BT} \tan(2mJ/k_BT)}{2mJe^{2mJ/k_BT} + 2mJe^{-2mJ/k_BT}}
\]

This has always the solution \(m = 0\). If the derivative of the right hand side is less than 1 at \(m = 0\), the are two additional non-zero solutions. Thus we have a phase transition when \(2J/k_BT = 1\) giving \(k_BT_c = 2J\). When the magnetisation approaches zero we may Taylor expand

\[
m = \tan(2mJ/k_BT) \approx \frac{mT_c}{T} - \frac{1}{3} \left( \frac{mT_c}{T} \right)^3 + ....
\]

Solving for \(m^2\) gives:

\[
m^2 = 3 \left( \frac{T}{T_c} \right)^3 (T_c/T - 1),
\]
and \( m \propto (T_c - T)^{1/2} \) and thus the exponent \( \beta = 1/2 \).

2. 

\[
f'(m) = b(T - T_c)m + cn^5
\]

\( f'(m) = 0 \) gives the minima at \( m = 0 \) for \( T \geq T_c \) and \( m = \pm (b(T_c - T)/c)^{1/4} \) for \( T \leq T_c \).

Close below the phase transition, we thus have the free energy:

\[
f(m, T) = f(\pm (b(T_c - T)/c)^{1/4}, T) = -\frac{b^{3/2}}{3c^{3/2}}(T_c - T)^{3/2},
\]

giving the entropy

\[
S = -\partial f/\partial T = -\frac{b^{3/2}}{2c^{3/2}}(T_c - T)^{1/2}
\]

and the heat capacity

\[
C = T\partial S/\partial T = T\frac{b^{3/2}}{4c^{3/2}}(T_c - T)^{-1/2}.
\]

Thus the exponent \( \alpha = -1/2 \) when approaching the phase transition from below. If we approach the phase transition from above, the magnetisation is zero all the time and the exponent therefore equal to 0.

3. Using

\[
f(t, h) \propto t^{-d} f(l^n t, l^n h)
\]

we can express the exponents \( \alpha, \beta, \gamma \) and \( \delta \) in terms of \( y_t \) and \( y_h \). We have

\[
m(t, h) = -\frac{\partial f}{\partial h} = l^{y_h - d} m(l^n t, l^n h)
\]

and

\[
\chi(t, h) = \frac{\partial m}{\partial h} = t^{2y_h - d} \chi(l^n t, l^n h).
\]

Choose \( h = 0 \) and \( l = |t|^{-1/y_t} \) and we get

\[
m(t, 0) \propto |t|^{-(y_h - d)/y_t} m(\pm 1, 0) = |t|^{\beta} \text{ which gives } \beta = (d - y_h)/y_t
\]

Similarly we get

\[
\chi(t, 0) \propto |t|^{-(2y_h - d)/y_t} \chi(\pm 1, 0) = |t|^{-\gamma} \text{ which gives } \gamma = (2y_h - d)/y_t
\]

and

\[
f(t, 0) \propto |t|^{d/y_t} f(t/|t|, 0) = |t|^{d/y_t} \text{ which gives } c \propto |t|^{d/y_t - 2} \equiv |t|^{-\alpha} \text{ and } \alpha = 2 - d/y_t.
\]

This gives

\[
\alpha + 2\beta + \gamma = 2 - d/y_t + 2(d - y_h)/y_t + (2y_h - d)/y_t = 2
\]

Next, we have

\[
m(0, h) \propto |h|^{-(y_h - d)/y_h} m(0, \pm 1) = |h|^{1/\delta} \text{ which gives } \delta = y_h(d - y_h).
\]
Using the relations above we can express \( y_h \) in terms of \( \beta \) and \( \delta \) and obtain then finally

\[
\beta(\delta - 1) = \gamma
\]

4. The Hamiltonian for an N spin system is

\[
H = -J \sum_{i=1}^{N-1} S_i S_{i+1} = - \sum_{i=1}^{N-1} J_i S_i S_{i+1},
\]

With \( J_i = J \). This gives the partition function:

\[
Z = 2(2 \cos \beta J)^{N-1} = 2 \Pi_{i=1}^{N-1} (2 \cos \beta J_i).
\]

Order the indices’s \( i \leq j \leq k \leq l \) and we can compute the four spin average by inserting additional factors \( S_i S_j = 1 \) as:

\[
\langle S_i S_j S_k S_l \rangle = \frac{\text{Tr}(S_i S_j S_k S_l e^{-\beta H})}{Z} = \frac{1}{Z} \text{Tr} \left( S_i (S_{i+1} S_{i+1}) \ldots (S_{j-1} S_{j-1}) S_j S_k (S_{k+1} S_{k+1}) \ldots (S_{l-1} S_{l-1}) S_l e^{\beta \sum_{m=1}^{N-1} J_m S_m S_{m+1}} \right) = \frac{1}{Z} \frac{\partial^{j-i+k-l} Z}{\partial J_i \ldots \partial J_{j-1} \partial J_k \ldots \partial J_{l-1}} = (\tanh(\beta J))^{l-i+l-k}
\]

5. See Bergersen & Pischke p. 94-96 (3.10).