1. Derive a general equation that relates the mean square fluctuations in energy to the heat capacity in a canonical system (constant temperature). Show how the relative energy fluctuations go to zero when the system size goes to infinity! (1 p.)

2. The Bragg-Williams approximation gives the Gibbs’ free energy for the Ising model as a sum of a mean field energy and an entropy of an ideal mixture

\[ G(m, h, T) = -\frac{a J N}{2} m^2 - N h m + N k_B T \left[ \frac{1 + m}{2} \ln \frac{1 + m}{2} + \frac{1 - m}{2} \ln \frac{1 - m}{2} \right]. \]

Derive from this an equation that determines the magnetisation, \( m \), in the mean field approximation! Show that without an applied field, this equation has only one solution \( m = 0 \) above a certain temperature and determine this temperature. Discuss qualitatively what solutions one has depending on temperature and applied field. (2 p.)

3. Consider a Landau model with free energy

\[ F(\phi, T) = \frac{a(T - T_0)}{2} \phi^2 - \frac{c}{3} \phi^3 + \frac{b}{4} \phi^4, \]

were \( \phi \) is the order parameter while \( T_0, a, b \) and \( c \) are positive constants. Show that this gives rise to a first order phase transition and determine the transition temperature and the jumps in order parameter and heat capacity at the transition temperature. (2 p.)

GOOD LUCK!
\[ \text{Var}(E) = \langle (E-E)^2 \rangle = \langle (E-E)^2 \rangle = \langle E^2 - 2E\langle E \rangle + \langle E \rangle^2 \rangle = \langle E^2 \rangle - 2\langle E \rangle \langle E \rangle + \langle E \rangle^2 = \langle E^2 \rangle - \langle E \rangle^2 \]

By definition: \[ \langle E \rangle = \frac{\text{Tr} e^{-\beta E}}{\text{Tr} e^{-\beta E}} \]

\[ C_V = \left( \frac{\partial}{\partial \beta} \left( \frac{\partial \langle E \rangle}{\partial T} \right) \right) = \frac{2}{T} \frac{2}{\partial T} \langle E \rangle = \frac{2}{k_B T^2} \frac{\partial}{\partial T} \langle E \rangle = \frac{1}{k_B T^2} \frac{\partial}{\partial T} \langle E \rangle \]

\[ \frac{\partial \langle E \rangle}{\partial \beta} = \frac{2}{k_B T^2} \left( \frac{\text{Tr} e^{-\beta E}}{\text{Tr} e^{-\beta E}} \right) = \frac{\text{Tr} e^{-\beta E}}{\text{Tr} e^{-\beta E}} \left( \frac{\text{Tr} e^{-\beta E}}{\text{Tr} e^{-\beta E}} \right) = \frac{\text{Tr} e^{-\beta E}}{\text{Tr} e^{-\beta E}} \left( \frac{\text{Tr} e^{-\beta E}}{\text{Tr} e^{-\beta E}} \right) = \frac{\text{Tr} e^{-\beta E}}{\text{Tr} e^{-\beta E}} \left( \frac{\text{Tr} e^{-\beta E}}{\text{Tr} e^{-\beta E}} \right) \]

\[ = \langle E^2 \rangle - \langle E \rangle^2 \]

\[ \Rightarrow \quad \langle E^2 \rangle - \langle E \rangle^2 = \text{Var}(E) = k_B T^2 C_V \]

Relative energy fluctuations:

\[ \frac{\text{Var}(E)}{E} \sim \frac{N}{N} \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty \]

as \( E \sim N \) and \( C_V \sim E \sim N \) (extensive)
2. See ch. 3.2 (p.67-68) in P&B!
\[ F(\phi, T) = \frac{a(T-T_0)}{2}\phi^2 - \frac{c}{3}\phi^3 + \frac{b}{4}\phi^4 \]

\((T_0, a, b, c \text{ are constants} > 0)\)

Qualitatively we have the following behavior of \(F\) as a function of \(\phi\) for different temperatures:

- **\(T_0 < T < T_c\)**: 
  \[\phi \to 0 \text{ defines a global min. of } F\]

- **\(T = T_c\)**: 
  \[\phi = \phi^* \text{ and } \phi = 0 \text{ define global min. of } F\]

- **\(T > T_c\)**: 
  \[\phi = 0 \text{ define a global min. of } F.\]
  We have also a metastable min. that vanishes for \(T > T_c\).

At \(T = T_c\) we must have

\[
\begin{align*}
\frac{\partial F}{\partial \phi} &\bigg|_{T=T_c} = 0 = a(T-T_0)\phi^* - c\phi^*^2 + b\phi^*^3 \\
F(\phi^*, T_c) &= F(0, T_c) = 0 = \frac{a(T_c-T_0)}{2}\phi^*^2 - \frac{c}{3}\phi^*^3 + \frac{b}{4}\phi^*^4
\end{align*}
\]

(1)

\(\frac{\partial^2 F}{\partial \phi^2} \bigg|_{T=T_c} < 0 \Rightarrow \phi = 0\) and \(\phi = \phi^*\)

\(\Rightarrow T_c = T_0 + \frac{c}{4ab}\)

(1) \(\Rightarrow \phi^* = 0\) and 
\[a(T_c-T_0) - c\phi^*^2 + b\phi^*^3 = 0\]  

(2) \(\Rightarrow \phi^* = 0\) and 
\[\frac{a(T_c-T_0)}{2} - \frac{c}{3}\phi^*^2 + \frac{b}{4}\phi^*^3 = 0\]  

(3) and (4) \(\Rightarrow \phi^* = \frac{2c}{3b}\) and \(T_c = T_0 + \frac{2c^2}{9ab}\)

So at \(T = T_c\) the order parameter jumps from \(\phi = 0\) to \(\phi^* = \frac{2c}{3b}\) (1st order transition)
The jump in the heat capacity is found from the definition of
\[ C = -T \frac{\partial^2 F}{\partial T^2} = -T \frac{\partial^2 F}{\partial T^2} \]
\( (T_0, a, b, c \text{ are temp. indep. const.}) \)
\[ C = -T \frac{\partial^2 F}{\partial T^2} = \ldots = -\frac{T_0}{2} \frac{\partial}{\partial T}(\phi^2) \]

Use \( T \to T_c^+ \) we have \( \phi = 0 \) \( \Rightarrow \) \( C = 0 \)

Use \( T \to T_c^- \) we have \( \phi = \phi^* \) \( \Rightarrow \) \( C = -\frac{T_0}{2} \frac{\partial}{\partial T}(\phi^2) \) \( \mid \phi = \phi^* \)
\[ = \ldots \text{ (a quite nasty expression)} \]

This is then the sought after jump in \( C \) at \( T = T_c \)